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The Journal of the Mathematical Association

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THE MATHEMATICAL ASSOCIATION

AN ASSOCIATION OF TEACHERS AND STUDENTS OF ELEMENTARY MATHEMATICS

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'I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.' BACON

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Presidential Address to the Mathematical Association

BY L. D. ADAMS

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The publication of two reports on popular education does mark an epoch in the history of the Association. Our membership includes a relatively small number of primary and secondary modern school teachers, since the number of persons specially qualified in Mathematics in those schools, though increasing, is still small. We have therefore offered advice in two fields in which as actual teachers most of us are not concerned. There are however two other groups of members who have direct contact with these schools. It is significant that there were four members (including myself) of both primary and secondary modern school sub-committees who belonged to these groups, namely two training college lecturers (or ex-lecturers) and two inspectors (or ex-inspectors) of schools. Lecturers in training colleges and the inspectorates, local and national, do indeed form vital links between persons who are generally of specialist education and those who are concerned with education as a whole, either

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because their pupils are young or because they are still so intellectually immature that they are best served by modes of education which are not too highly specialised.

May I be unashamedly personal for a few minutes? My own experience of education divides itself neatly into two halves. In the first half as pupil and as teacher in various schools and colleges I experienced almost every form (other than technical) of selective education. During the second half by my own choice I have been immersed in popular education carrying this interest on through contacts with training colleges during ten years of retirement. For much of this later half of my life I functioned as jack-of-all-subjects and I have even tried to conceal the fact that I viewed education with the eyes of a teacher trained as a mathematical specialist.

I am grateful for this non-specialist experience which cured me of many of the mistakes with young children to which those versed in teaching older pupils are liable. It gave me a lot to learn in middle life, in fact a second cycle of personal experience. I can recommend observation of young children as a refresher course, with the warning that to observe as an aunt or a parent does not give a sufficiently random sample of children to serve as substitute for a wider field of observation.

Such mathematical comment as has survived this non-specialist experience has been put at the disposal of the two sub-committees on which I have served. What then can I say to you now? After searching the annals of the Association I can find no more than passing reference to the influences which have been shaping popular education. Past Presidents, notably the late Mr. Siddons whose account of the early work of the Association reported in the Gazette of February 1936 is a classic, have dealt fairly fully with influences on selective education and with the history of various aspects of Mathematics.

I am no historian, but I had the good fortune to begin my observation of elementary education, as it was then called, at a critical period, 1925, when powerful new influences were at work and receiving administrative recognition but when many schools still retained clear evidences of their Victorian traditions. This period and ensuing years are well-documented for education as a whole, but I shall try to pick out some mathematical aspects of changes and add to them my personal impressions as an ordinary grammar school teacher of Mathematics plunged into what was in 1925 a startlingly different educational world.

It is convenient to consider influences on popular education in two categories; firstly the purely educational developments arising from increasing knowledge of young children and how the immature

learn, and secondly administrative changes, some favourable to educational development, for instance reduction in the size of classes, and others only measures of expediency which may even be adverse to it.

It is natural that educational research should be rich in its study of the youngest children especially in the mathematical field, for even at infant level thought processes are immensely complex. Fortunately infant schools have been comparatively free from administrative interference though sharing in administrative advances and teachers in them have been well enough informed and supported to make good use of educational theory. In the 1920's infant schools already had an established reputation and mathematically the best of them had learned many of the lessons of Froebel and Montessori. Moreover these intellectual advances had gone on side by side with advances in the treatment of young children as whole persons, supported by knowledgeable or at least sentimentally indulgent public opinion. This outside support had had administrative backing. The establishing of School Boards in 1870 had brought into contact with the schools persons of goodwill and good education. As an example my own grandmother served on the School Board of a Midland city from 1871 and it was written of her at her death in 1894, 'It was to her very largely that thousands of young children . . . in the infant schools . . . were indebted for having their infantile years made happy by the introduction of kindergarten methods. Being a woman and a highly educated one (she left school at fifteen), with deep and keen insight into child nature she was not trammelled by *preconceived notions* . . . she brought to bear on the schools an insight and intelligence that was extremely valuable because it was not merely official . . .'. This is a favourable picture of outside influence, but I could name from another personal source a teacher who was dismissed in 1888 for refusing to keep in after school five year olds who did not know the multiplication table expected of them. Her reason for refusing was that in her class of between 70 and 80 infants these children were already dropping off to sleep and falling from the classroom's galleries.

I think that when I first visited infant schools they had already passed one of their climaxes of development mathematically. The teachers' adult need of a systematic framework was imposing itself too rigidly on what children were given to do, there was a multiplicity of formal apparatus and a tendency to assume that if a little child had been through progressive stages of teaching he had learned what he had been taught. Influences from upper departments were strong and I well remember one whole area in which infant schools were acquiring merit with them by teaching the children to do addition of hundreds, tens and units before promotion. They did

this by the simple devices of adding numbers in the columns by making and counting strokes and then using a remembered routine for passing from column to column. Unfortunately junior teachers did not realise that ability to get answers right argued no knowledge of addition tables or understanding of how we write numbers. The lack of experience of using small numbers was clearly traceable at fourteen years of age in the slowness of using addition and subtraction bonds and there must have been all sorts of difficulties for the less able arising from the lack of groundwork in decimal notation. On the other hand I remember an infant school so obsessed by the power of decimal notation that it was imposing on children methods of dealing with small numbers which were logical but questionable as mental habits and emphasising this by elaborate apparatus which must have impeded automatic response. I was at the time quite taken in by the beautiful sequence of the teaching.

Just before and during the second world war the expansion and progress of nursery school work again threw emphasis on balanced child development. The good sense of teachers and the support of such organisations as the Froebel Foundation carried work with younger children again into the advance guard of primary education. Mathematics teaching tended to lag behind other aspects of advance. The subject was still referred to as 'one of the skills', but it was beginning to be thought of as something much more than this with skill in reckoning to be encouraged, but as a bi-product of use and the servant of need.

The work of Piaget and others into the nature and timing of mathematical concepts is thus being welcomed as support for teachers' own growing realisation that maturation must have its way and its time. I think that at present the idea that the maturation of the individual sets limits to fully effective teaching is more prominent than the more constructive aspect that it makes demands on ways of working. We await this interpretative stage in which we find out in terms of children as we know them how best to help mathematical ideas to take healthy root. At present there is perhaps some danger that interested teachers may mistake instruments of research, necessarily somewhat artificial and limited, for instruments of teaching. With the youngest children all that may be needed is the appropriate intervention of the adult in everyday happenings, but as the curriculum becomes more systematised more and more does the teacher have to create suitable opportunities for new mathematical learning.

The findings of contemporary research of course extend far beyond the infant school, though we may as yet be unready to heed more than the quite general implications about stages of learning and rates of maturation.

The influence of administrative changes is not a dominant theme in infant education but it is the best thread on which to hang the story of change in popular education for pupils of seven years of age and upwards. When I first visited elementary schools in 1925 they were on the eve of a great re-organisation which was both administrative and educational in intention, but schools above the infant stage were still organised in 'standards', a term which for 60 years had implied standards of formal attainment and which had been extended in number to cover raisings of the school age. In many schools rigid views about attainments led to wide ranges of age in some of the classes, as much as five or more of the seven years of school life from seven to fourteen.

It was a shock to me to find that what I had been taking for granted as a normal syllabus of arithmetic for pupils up to eleven years of age was so patently unsuitable for many boys and girls who were grovelling in lower standards at a late age. I think for some years I accepted wide differences of ability as the *only* explanation and expected the cure to lie *solely* in changes of organisation rather than in changes of approach. Constant contact with infant classes, however, helped me to see how the teaching of Mathematics might fit in to a more general theory of education for primary school stages, and some acquaintance with the history of 'standards' made me sceptical of the theories by which organisation by attainment without regard for age and interest was often bolstered up.

The name 'standard' goes back to before the days of universal popular education. In the Revised Code of 1862 the payment of grant to schools qualifying for it was based on the individual examination of pupils in the 3 R's at various levels called standards. Syllabuses were quite cut and dried: thus in arithmetic, Standard 3 worked a sum in any simple rule as far as short division (inclusive). Standard 4 a sum in compound rules (money). Standard 5 a sum in compound rules (common weights and measures). This pattern, with its emphasis on the dry bones of applied arithmetic, was designed when school life was very short. It is still visible in many primary schools today.

The examination for grant was administered by inspectors and their assistants. The effects on arithmetic teaching are summed up by Matthew Arnold,* who from the first disapproved of this Revised Code. In his report of 1868 he writes '. . . To teach children to bring right two sums out of three without really knowing arithmetic seems hard. Yet even here, what can be done to effect this (and it is not so very little) is done, and our examination in view of *payment*

* Reports on Elementary Schools 1852-1882. Matthew Arnold. H.M. Stationery Office (Eyre & Spottiswoode) 1910, page 128

by results cannot but encourage its being done. The object being to ensure that on a given day a child shall be able to turn out, worked right, two out of three sums of a certain sort, he is taught the mechanical rule by which sums of this sort are worked, and sedulously practised all the year round in working them; arithmetical principles he is not taught, or introduced to the science of arithmetic. . . . The most notable result obtained will be that which has been most happily described by my colleague Mr. Alderson. "Unless a vigorous effort is made to infuse more intelligence into its teaching *Government Arithmetic* will soon be known as a modification of the science peculiar to inspected schools, and remarkable for its sterility".'

Matthew Arnold indeed saw the teaching of arithmetic as a means of education and deplored the extinction of a like vision in a section of the teaching profession. We have not yet learned our lesson. If we are to go on examining the mathematically immature whether in primary or modern schools, we must learn how to test what matters in learning and not solely the mechanical end-points of it, which may or may not have been reached by desirable means.

The code of 1862 continued in force for some thirty years. I suspect that under it began the habitual use of incentives to correct computation such as the tawse and the cane. Its effects were perpetuated by the examination which accompanied the great increase in secondary school places at the beginning of the century. For a grant-paying machine was substituted a step-ladder for the ambitious. Able junior pupils improved their chances in the 'scholarship examination' by promotion into higher standards at the expense of the less successful, and generally tended to set the pace for these standards and for the examination. Also a vicious circle to which examinations are liable had begun. Questions were set because such work was being done in the schools and the schools went on doing the work because of the examination. I do not wish to harp on examinations, but these have been one dominant influence on primary teachers especially in Mathematics which appears so deceptively easy to test. I was told only the other day of a primary school master who was exceptionally knowledgeable about the possibilities of primary school work in Mathematics but who was afraid, perhaps mistakenly, to make changes, because of the selection test in his area which put a high premium on speed in mechanical work.

It took the re-organisation of schools with the break at eleven following the Hadow Report of 1926 to focus attention on the needs of primary pupils of all types. Primary schools were to be stabilised by a top age group which passed to a Senior or to a Secondary School leaving none behind, however low their attainment. The

many unreorganised schools remaining (and some remain to this day) gradually accepted a policy of keeping their top classes for older pupils.

Although this break at eleven is not universally agreed as the right age for change of school it has some great advantages. If a later age were chosen to end the comprehensive primary stage, small schools would have great difficulty in coping with individual differences within age groups in Mathematics. The range at eleven is already too great for some small or weakly-staffed primary schools to make good enough provision for the ablest pupils. The really precocious are seldom catered for.

The educational theory of primary education, which was elaborated in a second Consultative Committee report in 1931, is often stated shortly as learning from activity and experience. It was least well understood in its application to Mathematics. Attempts to give juniors things to do, often at too low a level of experience, and to let them 'progress at their own rate' in arithmetic (a slogan of the 1930's) were so often time-wasting that they brought the theories themselves undeservedly into disrepute. I was told only the other day of a girl just transferred to a grammar school who was surprised at the fact that she had mathematics *lessons*. Her progress in arithmetic in a small primary school had been dominated by a text book through which she worked *at her own rate* with occasional help. Small wonder arithmetic was not her favourite subject. Arithmetic text books even when garnished with a few pictures are amongst the dreariest of school books. They deal only with limited aspects of learning the subject; even the best of them cannot provide a substitute for discussion with the teacher and fellow pupils or deal adequately with the need for experiment and experience.

However, after re-organisation, ways of dealing with differing needs within classes gained ground; group work in arithmetic (more easily organised in schools where there is no homework) became quite usual even though the size of classes was reduced but slowly. (In 1926 the legal limit for a class was sixty but I actually came across a class with seventy-three on its books as late as October 1928.) Again many small infants' schools were, on re-organisation, absorbed into junior and infant departments and, although this was perhaps a temporary set back to infant education, there was an over-all gain in understanding the change from infant to junior classes with, as I see it, a gradual and encouraging improvement in the mathematical aspects of school work in the vital period six to nine years of age. There is still far to go; this period is a great time for exploration and we are still in too great a hurry to tell children how to work things in our way. Time does not allow me to give you examples of the wonderful original thinking which may occur when

a quite ordinary child has had the luck to meet a situation *before* being taught the adult rule for dealing with it.

All types of school except nurseries suffered great setbacks during the years of the second world war, but in primary education there was progress as well as setback. Examples spring to mind both of outstanding success in bringing the learning of Mathematics into the full stream of experience and understanding, and also of its antithesis, a horribly mechanised approach with its accompaniment of unbelievable howlers when memory failed.

From my now limited field of observation I should say that primary education in Mathematics is steadily gaining both in mathematical scope and in the degree of understanding behind work in number. But, the art of reckoning is still in popular estimation the *only* evidence of progress to be sought, and soundness in this art is impeded by obscuring fundamentals and usefulness under pile upon pile of rules.

As teachers we surely ought to try to bring up our pupils to form a public opinion which is better educated about Mathematics, especially about these early stages which are within the understanding of any person of good general education (even if they are slow computers).

The war years bore more heavily on the mathematical education of older pupils which had been making great progress during this century up to 1939. There had been in addition to the increased selection for grammar schools much concern for older scholars in the elementary schools particularly for those abler pupils that we might now call 'A-stream modern'. The curriculum had been widened for all, and even before Hadow re-organisation Local Education Authorities had been encouraged to try out various ways, including selective central schools, of extending the work for the ablest elementary pupils. Mathematical advances had been mainly towards teaching at a slightly later age Mathematics as taught in secondary schools with perhaps, for boys, developments of a practical kind in geometry and mensuration. Girls often missed geometrical experience and therefore suffered even more than boys from that most virulent disease, still I fear prevalent today, the learning of techniques of algebra *by rote* without either clear purpose or adequate background knowledge of number. This sort of superficial teaching also arose in the elementary schools when teachers used with pupils still at school the syllabuses they would meet again in the evening institutes. Where separate schools with some technical, trade or commercial bias existed Mathematics was treated as a purposeful tool, but bias was sometimes carried too far and spoiled the balance of the subject as an instrument of education.

The Senior Schools which followed Hadow re-organisation and which flourished in the 1930's had to face a wider problem, the teaching of *all* grades of ability remaining under the elementary code. They were indeed Secondary Modern Schools in all but name, privilege and the one year of age, fourteen to fifteen. They were for the most part staffed by teachers whose experience was with the children of the elementary school. Many teachers had themselves been so educated up to fourteen, so that they understood their pupils' background as though they were fellow scholars. Many of them had a real interest in Mathematics even though not highly qualified; some were such as might today have entered teaching through a University. To these old hands were added young teachers interested in a new venture and perhaps better qualified as subject specialists. The combination was often a strong one.

I could illustrate, in approach if not in detail, from what I saw in these schools most of the recommendations contained in our recent modern school report; in particular the best of these schools tried to suit their curriculum to their pupils' world: to the rural community, to the life of an industrial town or to the kind of folk living in a dormitory area of a city. One girls' school in a vicious quarter of a great sea-port became famous for the way it enlisted the co-operation of girls who certainly had no interest in learning for its own sake and who, whatever their ability or the scholarliness of their teachers, would have spurned a merely conventional approach to Mathematics. Such an extreme example casts a spotlight on an essential difference between compulsory education and a chosen school career.

The number of pupils who regard Mathematics as of importance to their future, or enjoyable in itself, is increasing, but there still remains a majority who, in the words of the modern school report, must be helped 'to see Mathematics as a subject that touches their lives and is worth their attention.' Recent discussions in the press about further raising of the compulsory school age make me feel more strongly than ever that for ordinary children at least, we need to take so-called applied mathematics out of cold storage, to include in it mathematical usages of any and every sort and to re-think how we treat such usages as vehicles both of general and of mathematical education.

I think that scant justice has been done in our thoughts to the pioneer efforts of Senior Schools and indeed of the top classes of many unreorganised schools of the pre-war period. The best of them took advantage of their freedom and showed flexibility and powers of adaptation controlled by a knowledge of circumstances which was the fruit of long experience.

Unhappily Mathematics and Science in Senior Schools were

among the earliest war casualties in education, especially in boys' schools. Almost every teacher with any pretensions to good qualifications in Mathematics who survived a call to the services was transferred to teaching in a Grammar School. The generation of wise rather than highly qualified men and women who had been the cream of the old elementary school were nearing the end of their service and the burden which fell on them was very heavy. New teachers coming into the service were those who had passed to grammar schools at eleven and who, for all their advantages, had less in common with their pupils.

I consider it a tragedy that the natural evolution of mathematics teaching through the development of Senior Schools was almost obliterated by the war. It cannot be revived in spirit without enormous effort, if only because of the conservatism of our profession. The 'set' of our minds is inevitably to teach how and what we were ourselves taught. Our idea of elementary mathematics is coloured by what we learned at eleven, twelve and thirteen, but our mode of learning is not necessarily the best mode for our pupils who are in a different generation and may be in very different categories as thinkers.

The Modern School, cut off from its mathematical history by the war, is thus extraordinarily susceptible to the influences of today some of which are far from educational. Questions of prestige and expediency may lead schools to try to acquire a veneer of success. To me the most hopeful tendency is the upsurge of primary education in Mathematics. It seems to me most appropriate that our report *Mathematics in Secondary Modern Schools* should be so definitely a sequel to the 1955 report *Mathematics in Primary Schools*.

When I drafted this address I tried to avoid except for incidental references one of the main means of influencing popular education, one which completes the cycle of influence, namely the education and training of teachers. The Association has not yet reported on this subject, though there has been a sub-committee discussing it. Very great changes arising from the extension of the two-year course in colleges to three years are imminent and this is not a moment to make detailed comment; but what I have said about the mathematical staffing of schools makes some comment almost necessary as a conclusion.

Interesting changes in the organisation of the education and training of teachers have gone on side by side with the changes in popular education and I can at least speak of these as one who has had the good fortune to be linked with the work of the colleges in one capacity or another continuously since 1926.

In 1926 only a few teachers in elementary schools for pupils from

five to fourteen years were subject specialists, and few of these had any mathematical qualification other than perhaps Mathematics as one of several subjects in the Teachers' Certificate. Almost all teachers taught arithmetic and a problem of colleges was, as indeed it still is in dealing with primary school work, that of coping with a wide range of attainment, interest and aptitude amongst students. Wide differences existed (as they do today) even amongst those who elected Mathematics as a special subject.

In 1926 the certificate examination was still conducted by the Board of Education. As a junior member of a panel of inspectors entrusted with the examination of Mathematics and coming to this task from teaching in a grammar school, I was immediately struck by the difficulty of setting papers which were neither soft options for the apt well-prepared candidates nor dreary drudgery for the less able but not unpromising future teachers. Inspectors were very interested in this work and in trying to keep some balance between the needs of the schools they visited and the personal education of the teacher, but I do not think that at the time of which I speak the examination was keeping pace with changing conditions in the colleges. The pre-college education of students was changing very rapidly and the gap between student and student, not least in Mathematics, was widening, in spite of, or even because of, the fact that the 1920's saw the last of the pupil teacher centres devised to deal with the fourteen-year-old school leaver.

Finally for the record, let me note what is familiar to so many of you. The certificate examination by the Board of Education was discontinued in favour of a decentralised system administered by Joint Boards under the auspices of the Universities and later by the Institutes of Education now in operation.

It would be impertinent for me to comment on all that the colleges have done for the schools under these systems and on the courageous experiments tried, including such temporary measures as the Emergency Training Scheme and the longer term but now inevitably temporary measure, the third year Supplementary Courses for two year trained and emergency trained teachers. I hope, however, that those of you who are grappling with the problems of Mathematics in the colleges, together with the many who are teaching their future students in the grammar schools, will not mind my emphasising what I think to be important however it may be accomplished.

The problem of making the most of the talents of pupils and students who, as things are, may have little interest in Mathematics but will have to teach it in its fundamental stages, is as important for the health of the subject as turning out more specialist teachers; indeed in the educational cycle it is an important step towards the making of more mathematicians. Moreover amongst those taking

Mathematics as a special subject in college there is great scope for dealing with differences of interest and aptitude. Popular education in Mathematics at its various levels can absorb without detriment a very wide range of talent in its teachers, provided that such mathematical knowledge as they have, however elementary, is happily based on genuine learning and is fundamentally sound.

The Small House, Oxford Street, Eddington, Berks. L. D. A.

THE CAMBRIDGE MATHEMATICAL TRIPOS

By W. F. Bushell

The last examination of the Cambridge Mathematical Tripos, when the candidates were placed in each class in Order of Merit and not in alphabetical order, was held in 1909. Last year was therefore the fiftieth anniversary of the end of a famous epoch. The University Calendar used annually to reproduce all the lists from 1747, the year when they were first printed, and they made fascinating reading, well known to mathematicians. Of course candidates were examined before that date, but full details have not been preserved. It was earlier called 'The Senate House Examination', and not before the start of the Classical Tripos in 1824 did the name gradually change.

Probably only a small minority of our Association took this Tripos prior to 1910, and hence many will not realise the interest taken in each list, and the publicity accorded to it by the Press. The name of the Senior Wrangler was always published with brief biographical notes, and similar attention was paid to some of the other Wranglers.

I was brought up to view these lists with interest. My father had been a high Wrangler in 1861, and indeed was subsequently a founder member of our Association in 1871. His son, grandson and great grandson are all now members after taking the Tripos under various regulations. Further my grandfather was a Wrangler in 1828, and other relations were at times in the list. This accounts for my interest in the development of the examination.

I suppose my experience was characteristic of many men of the time. I spent three years studying Mathematics for the Tripos of 1906. It was a queer distorted education, as although we were taught Electricity, Optics and other subjects now generally classified as Physics, I had never entered a laboratory at school or University! I made an attempt to remedy that deficiency in my fourth year by taking three subjects in the Natural Science Tripos. I wished to take Chemistry, Physics and Geology, but my tutor strongly advised me to substitute Botany for the latter subject 'because', he cynically observed, 'we always say here that the

easiest degree at Cambridge is an aegrotat in Botany'! Of course things have now changed. But some science subjects even to-day are surely harder than others.

The need for practical demonstration was not always recognised. The story is told of how Clerk Maxwell at the Cavendish laboratory, long before my time, had just completed some experiments in Conical Refractions and invited Todhunter, the famous mathematical coach to come and see them. 'No', said Todhunter, 'I have been teaching Conical Refraction in Physical Optics for many years, and it might upset me'. When the further suggestion was made that his pupils might like to do so, he added, 'If a young man will not believe his tutor, a gentleman, and often in Holy Orders, I fail to see what can be gained by practical demonstration'. This is reminiscent of what Galileo said in a letter soon after he had used his telescope in 1610. 'My dear Kepler', he wrote to the great Astronomer, 'What would you say of the learned here who replete with the pertinacity of the asp have steadily refused to cast a glance through the telescope? What shall we make of all this? Shall we laugh or shall we cry?'

Todhunter was Senior Wrangler in 1848 and had acted as my father's tutor at St. John's. He was the author of many well known textbooks both elementary and advanced. In quite early days I used his Euclid and Algebra, now long out of date, but probably they were an advance on their predecessors. The dictionary of National Biography tells us that 'besides being a sound Latin and Greek Scholar, he was familiar with French, German, Spanish, Italian, and also Russian, Hebrew and Sanscrit!' Lighter stories were told of him. He had once been down to the river to see the May races. He never went again, on the grounds that he could visualise in his college room exactly what was happening, and considered his presence unnecessary.

It was told of him too that, unlike many mathematicians, he was completely unmusical, and could only recognise two tunes, one of which was 'God Save the Queen' and the other wasn't. Indeed he could only recognise these two apart by the fact that in one case people stood up and in the other case they did not.

His lifelong friend was Mayor, another famous fellow of St. John's. On his death in 1884, the Cambridge Review asked Professor Mayor, well known to many Cambridge men, to write his biography. Mayor had a passion for completeness and accuracy, but no sense of proportion and in the first three instalments gave full details of the life and genealogy of three Dissenting Ministers who had educated Todhunter. In despair the Editor would allow no more to appear. Mayor and Todhunter were lifelong friends, yet it was said that only once had they visited one another's rooms.

Before his time Hopkins had been a famous coach and taught seventeen Senior Wranglers. After his time there came Routh who achieved phenomenal success during the period 1855–88. He was son-in-law to Airy, the Astronomer Royal. He examined both my father in 1861 and then my brother in 1893. During the intervening period his duties as a famous coach naturally precluded him from acting in this capacity. Later there were Webb and others. Webb came from Monmouth and his father died just before the second portion of the examination. His tutor did not tell him immediately, but met him on the steps of the Senate House directly after the conclusion of the final paper on the grounds that he could give no help by going immediately to Monmouth, and that such a procedure might ruin his chance of becoming Senior Wrangler, a distinction which he actually gained in 1872. At a later date he acted as coach to my brother-in-law W. N. Roseveare who was a high Wrangler in 1885, and subsequently a fellow of St. John's. Webb was one of my examiners in 1906 and found me with a pen in my hand only a very few seconds after we had been told to stop. With difficulty I persuaded him not to tear up my paper!

Roseveare also came from Monmouth, and was therefore on friendly relations with Webb. The latter was often rather truculent to his small class of three or four pupils. He tended to complain of the small amount of work they produced for his inspection, and one day exclaimed in indignation, 'If anyone wants to see the biggest fool in the University he had better send to me.' 'And would you go, Sir?' was Roseveare's gentle answer!

At the beginning of this century R. A. Herman was probably the best known private coach. He was Senior Wrangler in 1882, and in 1906, when I took the Tripos, had the two bracketed Senior Wranglers as pupils, and again in 1907 the first three Wranglers. With the passing of the Order of Merit in 1910 these famous coaches tended to disappear, but in earlier years those who aspired to the highest honours eagerly looked for the man who was reputed to be most successful with his pupils. The system was not universal and indeed discouraged in my college, but it certainly helped me. At the beginning of my second year I went to a Senior Wrangler who was said to have ruined his health by overworking for the Tripos and insisted on wearing a college cap, if not a greatcoat, in his rooms. This proved unsatisfactory, and I went on to another private coach who smoothed my path, and to whom I owe a very considerable debt.

The system of private tutors, paid of course by the pupil, was objected to on two grounds. Firstly, that a student was entitled to get all he needed from his own college instead of an outside coach. Secondly, it was supposed that a man became unduly dependent on his tutor whose great anxiety was a high place in the Tripos. At its

conclusion, bereft of this fatherly instruction, he was helpless and did not continue his mathematical studies. I never felt there was much force in this second objection, although once it was widely urged.

The Mathematical Tripos had an historical interest far greater than that of any other. It was a development from the oral examinations and disputations of the middle ages when men had to 'wrangle' seated on three-legged stools or tripods, giving rise to the word Wranglers, or first class men, and Tripos for the final examination. It has already been said that lists were formerly published from 1747 in the Cambridge Calendar, when the examiners seem to have started the custom of printing them, but it was not until about 1820 that the examination started to become reasonably severe in Mathematics. The Classical Tripos was founded about the same time, but for nearly thirty years no man might take it until he had gained honours in the Mathematical. Indeed as has been previously stated the latter name had therefore to be substituted for the older one, 'The Senate House Examination'. About the same time the number of examiners was increased from two to four. Two were styled Moderators, and the other two Examiners, the Moderators of one year becoming the Examiners of the next. This was done to ensure continuity. The former name dates from the year 1680 when they were appointed to conduct the "disputations" over which the proctors had previously presided. In consequence, up to a late date, it was customary to show the final list to the Senior Proctor the evening before it was read out in the Senate House.

For most of the nineteenth century the examination was held in January and not until 1882 was the date altered to the early summer, perhaps owing to the cold in the Senate House where it was always held. It is on record that once the candidates found the ink frozen at their desks! Other alterations in the nature of the papers date from this period, and the new form continued for many years. It consisted traditionally of the first and second four days separated by a gap of about ten days. The first seven papers were comparatively elementary, and the next seven were severe. Indeed men placed towards the bottom of the third class, called Junior Optimes, got few marks in the latter. Those in the second class were styled Senior Optimes. Both names had come down from the past, but in the first five printed lists from 1747 which have come down to us the Wranglers and Senior Optimes are not separated. The examination was called Part I and II for historical reasons in the new regulations of 1882, and the best men went on to take a more advanced course, being examined in a so-called Part III in the following January, but it was soon perceived that six months were insufficient, and from 1886 onwards the main Tripos was called Part I and the more advanced examination, renamed Part II, was taken a year later, but

only quite a few of the best men entered for it. Thus January finally disappeared as an examination month. This arrangement lasted until the abolition of the Order of Merit in 1909–10.

The Senior Wrangler of my year (1906) was an Indian, and so was the sixth. This was by no means the first time Indians had done well. Once and once only was there a woman Senior Wrangler. This occurred in 1890 when Miss Fawcett was placed 'above the Senior Wrangler'. In those days the actual title could not be assigned to her. She was a daughter of the statesman, Henry Fawcett, seventh Wrangler in 1856. It was humorously said, but actually it was not true, that the male Senior Wrangler of that year never really recovered from the shock. The announcement of the result, when the list was read out in the Senate House, was always an occasion, but it was a trying moment for the man who knew he could only expect at best to be a low Wrangler. There was no guarantee when the examiner would finish the first class, and start with the second! At the conclusion, after reading out the three classes, he always uttered the word "Women" and proceeded to read their names in the appropriate order, but the undergraduates below the gallery from which the results were announced, always shouted 'Ladies, Ladies' for some seconds before he could carry on! He then threw down the printed lists, and a swirling mob of undergraduates below battled for their possession!

I have already stated that it is interesting to look at these lists which used to be published complete in the Cambridge Calendar from 1747. Many distinguished names appear in them. Lawyers and judges have found Mathematics, which teaches exactitude, useful in their subsequent careers. There were very many who became clergy, and later raised the high intellectual standard always possessed by the church. Further a considerable number became Fellows of their College or teachers at School or University.

There was a good deal of College rivalry, but two Colleges, Trinity and St. John's, were easily first among the Senior Wranglers produced. Starting from 1747, when the printed lists first appear, the latter led with 46 Senior Wranglers up to the new regulations of 1882; Trinity had gained 36, but taking the complete list up to 1909 inclusive, after which the Senior Wrangler disappeared with the Order of Merit, Trinity had 56 and St. John's 55. The next College, Caius, had 14, and the other colleges were in single figures. But this is true only if *all* bracketed men are regarded as Senior. Counting one Senior Wrangler per annum from 1747–1909 inclusive St. John's would appear to have 53 and Trinity 50. It is remarkable that these two Colleges should have been so far ahead. I ought to add that I once persuaded the late Mr. A. Robson, a former President of our Association, and interested in statistics, to work out these results from a

Cambridge calendar. I have not verified them but I do not doubt they are correct.

1907 was an 'annus mirabilis' for Trinity as the College had the top five Wranglers, and three of the four bracketed sixth. Indeed by the first method of counting Senior Wranglers Trinity had seven in the four years 1903–6. St. John's great period was rather earlier; from 1843–50 the college had seven seniors out of the eight, including Adams, Parkinson, Todhunter and Besant all famous in their day; the first is best remembered as the discoverer of Neptune.

The special position which was assigned to the Senior Wrangler is also shown by his presentation for a degree before the Vice-Chancellor. He was given it first and alone, other candidates coming in small groups. At the other end of the list another special position was given to the candidate being presented, as he left the Senate House, with a large wooden spoon! No doubt too much importance was attached to a man's exact position on the list, the tenth Wrangler for instance, being considered a better mathematician than the one two or three places lower, and this led to the famous controversy of 1906–7. The majority of residents wished to abolish the Order of Merit and arrange names in each class in alphabetical order, as was the rule in practically all University examinations. My college tutor, Arthur Berry, who had been Senior Wrangler, was in favour of the change, but a minority were strongly opposed. That admirable lecturer, Coates of Queen's, asserted that the change could mean the end of Mathematics at Cambridge. He felt that the competition among the best men to become Senior did them good, and was also influenced by the traditional glamour surrounding the reading out of the lists in the Senate House. The minority were determined, and brought up non-resident voters to the University to record their opinions, as could be done in those days; but they suffered defeat in February 1907. Certainly the result is not publicized in the national newspapers to-day as it was in former times, but the fact that the percentage of candidates entering has diminished is due to the abundance of other Triposes. It has already been observed that the Classical Tripos did not start till 1824, and the importance formerly attached to Mathematics is shown by the rule, which existed for nearly 30 years, that no man might take it until he had gained honours in the Mathematical Tripos. After the middle of the century many other subjects were gradually introduced for a degree.

It was said that attempts were generally made, when necessary, by the examiners to help a brillant Classic to overcome this obstacle. The late Mr. A. W. Siddons, formerly a President of our Association, told the story in a recent article in the Gazette, of a Classic who was weak at mathematics and went to a coach. He asked him to choose 20 propositions of Euclid likely to be set so that he might learn them

by heart. Later the number was reduced to 10. After the examination the delighted undergraduate went to his coach's room and said 'I am through; I got 8 of the propositions you chose, and got them all right to a comma'. As a afterthought he added, 'I am not sure that I put the right letters at the right corners, but I suppose that does not matter'! I find it difficult to believe that so many Euclidean propositions were really set even in the period of 25 years prior to 1850, but the story does illustrate the attempts alleged to be made to help the brilliant Classic who happened to be a weak mathematician. The story may well be true of the Matriculation examination.

At the beginning of this century one proposition of Euclid was always set as the first question in the first paper, but the accompanying rider was not so easy. The last question in the same paper was taken from the first book of Newton's 'Principia', the usual plan being to ask the candidate to reproduce a piece of bookwork and then add a rider based upon it. Indeed most questions were modelled on these lines. Fortunately, Newton's Latin was not demanded!

I was asked in 1906 to 'enunciate Lemmas VI and VII, from Newton's 'Principia'. I should be sorry to attempt that to-day without reference to the text! But it is a tribute to his influence on Cambridge mathematics when we remember that the great book was published as long ago as 1687.

The seventh paper in each period of the four days was the Problem paper, a terror to many candidates! The problems were often much too difficult, and the examiners were supposed to have spent much time inventing them during the preceding months. It was said that satisfied examiners sent copies of the best to their friends to illustrate their inventive capacity. It would have been better if more attention had been paid to the powers of the ordinary undergraduate! Many subsequent Wranglers were satisfied if they solved two or three of the twenty or more that were set!

It must not be supposed that Science, apart from Mathematics, was entirely neglected at Cambridge in earlier days. Watson, for instance, was second Wrangler in 1759 and became Professor of Chemistry in 1764, though he had never read a syllable of the subject, nor seen a single experiment. He states he was tired of Mathematics and of Natural Philosophy and wanted to try his strength in a new pursuit. Later, as Bishop of Llandaff, he destroyed his work in Chemistry as a 'sacrifice to the notion of other people as to the occupations proper to the dignity of a church'.

In more modern days an examination in Natural Science was instituted in 1851, though it did not bring a degree until 1861. The first name in the former list in the Cambridge Calendar is that of G. D Liveing who had been eleventh Wrangler in 1850. Later he held the

post of Professor of Chemistry and, after retirement, lived at Cambridge until he was nearly 100. One of my happiest memories is that of dining at St. John's a few years after the First World War and sitting between him and T. G. Bonney, twelfth Wrangler in 1850, and subsequently an eminent geologist! I have never forgotten their natural and delightful courtesy to a young and inexperienced man! I even remember discussing with Bonney the site of the holy sepulchre in Jerusalem in which, for various reasons, both of us were interested.

No attempt has been made to consider how 'The Senate House Examination,' after 1824 to be the Mathematical Tripos, originated. Information can be gained from books by Rouse Ball and others. Nor has any attempt been made to appraise the value of the new regulations which often, during the past two centuries, changed the syllabus required. But it did seem suitable, fifty years after the cessation of the order of merit, to remind readers of what this famous Tripos meant to an earlier generation.

W. F. B.

ELEMENTARY DIVISION

By N. de Q. Dodds

I am convinced that many difficulties that arise in the Primary and Secondary schools would never do so if from the very beginning division were taught differently.

What I wish to urge is a complete break away from the traditional way of setting out division.

It is a problem of both the spoken and the written language, of explaining and of putting on paper what we can "do in our heads", or by using bundles of sticks.

Firstly, then, in the spoken language, I want to avoid the use of the word 'into', which I maintain is nearly always misused. For example, if 12 people miss the last train and find there are only 3 cars to take them home, the problem of division is not "3 into 12", but "12 into 3"—if a yard of ribbon is cut into four equal parts, finding the length of each part requires not "4 into 36" but "36 into 4."

So, avoid 'into' and say in these cases "12 divided by 3", and "36 divided by 4".

Secondly, in the written language, which is perhaps more important, let division be written $12 \div 3$ and then $\frac{12}{3}$, and emphasise that this is the reverse process to multiplication, and that $3 \times 4 = 12$; $4 \times 3 = 12$; $\frac{12}{3} = 4$; $\frac{12}{4} = 3$; are four ways of expressing the same truth.

Later, after suitably graded examples, use the same method for bigger numbers, showing the 'carry' figures in due course, but with special care that the figures in the result are directly underneath in the correct columns, thus:—

$$5\ 2\ 7\ \ 6 \div 4$$

$$= \frac{5^1\ 2\ 7^3\ 6}{4}$$

$$= 1\ \ 3\ 1\ \ 9$$

Now, practical work will show that division by 6, say, is the same as division first by 2 and then by 3, so that we may write,

$$\frac{24}{6} \qquad\qquad 712404 \div 42$$

$$= \frac{24}{2 \times 3} \qquad\qquad = \frac{712404}{6 \times 7}$$

$$= \frac{12}{3} \qquad\qquad = \frac{118734}{7}$$

$$= \ \ 4 \qquad\qquad = 16962$$

Remainders should be avoided until *after* the introduction of fractions, which will in fact be very much easier now.

Probably every child has had to share a bar of chocolate with another, fairly we hope, and will be familiar with the phrase "half each", so there will be no trouble about writing $1 \div 2$ as $\frac{1}{2}$ and calling this a "half", and similarly for the other simple fractions, and all the early work on fractions will easily follow, especially important being equivalent fractions, $\frac{3}{6}$, $\frac{2}{4}$, $\frac{4}{8}$, and $\frac{1}{2}$ all, now, obviously leading to the same result "half each".

After this should come $\frac{3}{2} = 1\frac{1}{2}$ and $\frac{7}{2} = 3\frac{1}{2}$ and many simple examples worked *both* ways, $1\frac{3}{4} = \frac{7}{4}$ as well as $\frac{7}{4} = 1\frac{3}{4}$.

Let remainders be written this way, and explain that in some problems the result can be left like this, as bars of chocolate can be broken into fractions, but if the problem were to share 29 conkers between four boys, we must write $\frac{29}{4} = 7\frac{1}{4}$ and give the result as "7 conkers each and 1 left over".

For division by factors arrange that at first there shall be no remainder to the first division, and then knowledge of the equivalent fractions will find the final remainder, if required. When there is a remainder to both divisions, the work described above on mixed numbers will make the final step of obtaining the remainder much clearer. The following examples illustrate this point:

$$\frac{2418}{42} \qquad\qquad \frac{2417}{42}$$

$$= \frac{2418}{6 \times 7} \qquad\qquad = \frac{2417}{6 \times 7}$$

$$= \frac{403}{7} \qquad\qquad = \frac{402\frac{5}{6}}{7}$$

$$= 57\tfrac{4}{7} \qquad\qquad = 57\tfrac{23}{42}$$

In the first case we know $\frac{4}{7}$ is equivalent to $\frac{24}{42}$ so we can say "remainder 24" and in the second case $3\frac{5}{6}$ is equivalent to $\frac{23}{6}$ and dividing this by 7 gives $\frac{23}{42}$ and so we say "remainder 23". The 'carry' figures would, of course, be put in only as long as is necessary.

This way, I am sure, is much better than learning a 'rule' which it is almost impossible for the child to understand, and all too easy to forget. The alternative suggestion that this work should be omitted has much to recommend it, because the method is, in fact, scarcely ever used later on.

Now, for 'long division', teach the use of the margin, or rather, the right-hand side of the paper, to show the detailed working, thus:

$$\begin{aligned} &593841 \div 29 \\ = &\frac{593841}{29} \\ = &\ 20477\tfrac{8}{29} \end{aligned} \qquad\qquad \begin{array}{r} 593841 \\ 58 \\ \hline 138 \\ 116 \\ \hline 224 \\ 203 \\ \hline 211 \\ 203 \\ \hline 8 \end{array}$$

This layout is exactly the style we shall require when, later on, logarithms are being used, and there need be no trouble over the placing of the figures in the result if the preliminary work has been carefully followed and if the inclusion of the zero has been introduced by such examples as $204 \div 2$, $612 \div 6$, $24112 \div 12$.

The advantages of these methods for later work are many. Reduction of fractions and cancelling are already learned, though not by these names; multiplication and division by fractions will now be much simpler to explain. For example, finding 2/3 of 6 means finding 1/3 and multiplying by 2, so we must write $\frac{6 \times 2}{3}$; and

division by a fraction can be written thus:—

$$3 \div \tfrac{4}{5}$$
$$= \frac{3}{\tfrac{4}{5}}$$
$$= \frac{3 \times 5}{\tfrac{4}{5} \times 5}$$
$$= \frac{3 \times 5}{4}$$

using here the familiar idea of equivalent fractions. We can of course leave out an intermediate stage when we have seen what happens every time. This is much better than learning the 'mumbo-jumbo' rule "turn it upside down and multiply", because the rule does not say why, and all too often the "it" that is turned upside down seems to be the wrong one, if both numbers happen to be fractions.

When algebra comes "think of a number, subtract 4 and divide the result by 7" will become at once $\dfrac{x-4}{7}$; and there will be less bother about $3x = 11$ leading to $x = \frac{11}{3}$ after the instruction "divide both sides by 3."

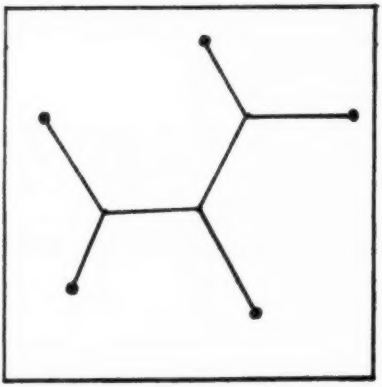
Haberdashers' Aske's Hampstead School. N. DE Q. D.

PROBLEMS OF ARITHMETICAL GEOMETRY[1]

L. MIRSKY

The subject matter of my talk is geometry in the sense that I shall deal with such entities as lines, circles, and squares. But the results I aim at bear little resemblance to the theorems of traditional geometry, which describe properties of given configurations. Our present object is quite different; we shall begin by specifying not configurations but processes (for example processes of selection), and then seek to determine how economically or how effectively these processes can be carried out. Questions of this nature are quantitative and the solution, if it is known, is generally expressed by a number. It is for this reason that I speak of 'arithmetical' geometry. The problems of the type I shall discuss are usually simple to state and require very little in the way of mathematical equipment for their understanding. However, their solution often presents formidable difficulties and is in many cases as yet far from complete. Let us now consider in some detail a few of the characteristic problems in this field.

[1] Lecture given to the Invariant Society, University of Oxford, in March 1958.



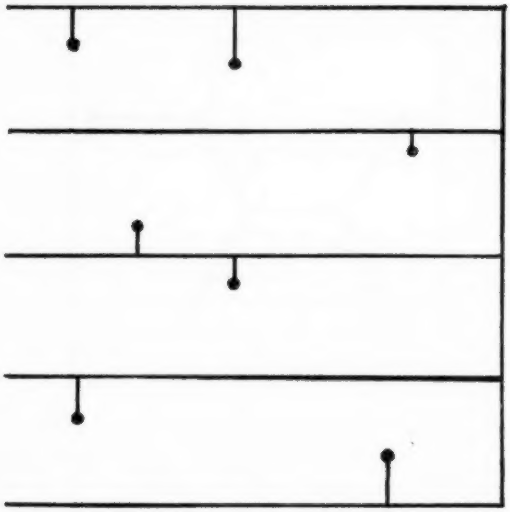
The network problem

Consider a city whose area is one mile square. It is proposed to build an underground railway, and its n stations are to be connected by a network of rails in the most economical manner. If the cost of the project is directly proportional to the length of the network, what kind of estimate can be given for it? It may be objected that the question, put in this form, is meaningless, since the estimate will depend on the distribution of the stations: if they are clustered within a small region the cost will be low, while if they are spread fairly uniformly throughout the area of the city the cost will be high. However, what we require, is an estimate of the cost which need not be exceeded whatever the distribution of the stations may be. To put it more mathematically, can we specify some number L_n such that *any* n points placed on a square of side 1 can be linked by a network whose total length does not exceed L_n? It is obvious that

$$L_n = (n - 1)\sqrt{2}$$

satisfies our requirements, but this number is unreasonably large and we naturally wish to choose L_n as small as possible. Let, then, l_n denote the smallest admissible value of L_n. This means that *any* n points on a unit square (no matter how placed) can be linked by a network of length $\leqslant l_n$; while, if $l < l_n$, then there exists *some* system of n points on the unit square which cannot be linked by a network of length $\leqslant l$. The problem now is to determine l_n.

Let q be a positive integer whose value will be fixed later. We proceed to construct a network in the following manner. Take three sides of the square and draw lines parallel to two of them and at a distance $1/q$ apart. The length of the network specified so far is



$q + 2$. Next, join each of the n given points to the nearest point on the horizontal lines in the diagram; the length of each segment drawn in this way is at most $1/2q$. Thus, we have produced a network joining the n given points, whose length does not exceed $q + 2 + (n/2q)$. Since this estimate is independent of the position of the points, we have

$$l_n \leqslant q + 2 + \frac{n}{2q}. \tag{1}$$

The choice of the number q is still at our disposal, and we take it to be the greatest integer not exceeding $\sqrt{(\frac{1}{2}n)}$. Thus

$$q = \sqrt{(\tfrac{1}{2}n)} - \theta, \qquad 0 \leqslant \theta < 1,$$

and so $n = 2(q + \theta)^2 \leqslant 2(q + 1)^2$. Hence, by (1),

$$l_n \leqslant q + 2 + \frac{1}{q}(q + 1)^2 \leqslant 2q + 5 \leqslant 2\sqrt{(\tfrac{1}{2}n)} + 5,$$

i.e.

$$l_n \leqslant \sqrt{(2n)} + 5. \tag{2}$$

This inequality means that n points on a unit square can always be linked by a network of length not exceeding $\sqrt{(2n)} + 5$. However, this is by no means the most precise statement that can be made. Indeed, it is possible to refine the above argument by considering, in the first place, two suitable networks and then choosing the

shorter of the two; this procedure enables us to effect a substantial improvement in the estimate (2) and leads to the inequality

$$l_n \leqslant \sqrt{n} + \frac{7}{4}. \tag{3}$$

This is as far as it is possible to go at present, but there is no reason to think that (3) is incapable of further improvement. It seems, for instance, likely that the limit, as $n \to \infty$, of $l_n/\sqrt{n}$ exists and that its value is $\geqslant 12^{-1/4}$, but this has not been demonstrated. Thus, although a promising start has been made, the network problem still remains unsolved.

The lid problem

We begin by recalling that the *diameter* of a figure F is defined as the upper bound of distances between all pairs of points in F. Thus the diameter of a circle is equal to its diameter in the usual sense, the diameter of a rectangle is equal to its diagonal, and the diameter of an astroid is equal to the distance between two opposite cusps.

To standardize the scale, we shall restrict our attention to figures of diameter 1; and we shall investigate the possibility of covering such figures by a circle. A circle which is capable (if suitably placed) of covering *any* figure of diameter 1 will be called a *circular lid*. What, then, can be said about the size of a circular lid? Obviously, any sufficiently large circle (say one of radius 20) is a lid. However, we are interested to know whether a more economic answer is possible, and we ask therefore: what is the radius of the *smallest* circular lid? If this radius is denoted by r, then a circle of radius r will cover *any* figure of diameter 1, while a circle of radius $<r$ will fail to cover *some* figure of diameter 1.

To determine r, let F be a figure of diameter 1 and assume, as may be done without loss of generality, that F is closed.[1] Let C be a circle of *minimal* radius which covers (i.e. contains) F,[2] and let γ denote the circumference of C. If no point of F were situated on γ, then a circle concentric with C but of smaller radius would still contain F,[3] and this would be contrary to our definition of C. Thus γ contains at least one point of F.

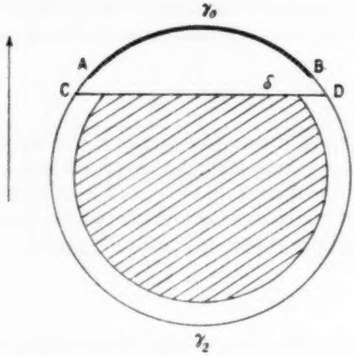
Assume, next, that all points of F on γ are contained within an arc γ_0 of angle[4] less than π. Take a second arc γ_1 on γ, having the same midpoint as γ_0, whose angle is greater than that of γ_0 but

[1] If F is not closed, we replace it by its closure.

[2] The existence of such a circle follows immediately by the properties of continuous functions.

[3] We use the fact that the distance between two closed disjoint sets is positive.

[4] By the angle of a circular arc we mean the angle it subtends at the centre of the circle.



less than π. Denote by δ the chord of C joining the extremities of γ_1 (so that δ is shorter than the diameter of C); by γ_2 the arc of γ complementary to γ_1; and by C' the segment of the circle C bounded by δ and γ_2. Since no point of F lies on γ_2, it follows that all points of F in C' lie within a segment of a circle concentric with C and having a smaller radius than C; this circle is represented in the diagram by the shaded area. If we now move C through a sufficiently small distance perpendicular to δ in the direction of γ_1, we shall obtain a new circle C^*, with the same radius as C, and containing F wholly in its interior. But this, as we have seen, is impossible; therefore, the points of F on γ cannot be contained within an arc whose angle is less than π. Hence one of the following two cases must arise.

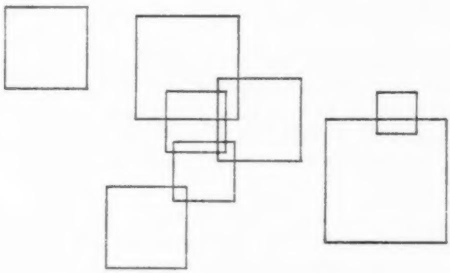
Case I. There are just two points X, Y of F on γ, and these points are the extremities of a diameter. In this case the radius ρ of C is equal to $\frac{1}{2}XY = \frac{1}{2}$.

Case II. There are at least three points of F on γ. Then three among them, say X, Y, Z form an acute-angled triangle. If $\hat{X}$ is its largest angle, then $\hat{X} \geqslant \frac{1}{3}\pi$; and so

$$\rho = \frac{YZ}{2\sin\hat{X}} \leqslant \frac{YZ}{2\sin\frac{1}{3}\pi} = \frac{YZ}{\sqrt{3}} \leqslant \frac{1}{\sqrt{3}}$$

Thus, in either case, we have $\rho \leqslant 1/\sqrt{3}$; and this implies that a figure of diameter 1 can always be covered by a circle of radius $1/\sqrt{3}$. Furthermore, the number $1/\sqrt{3}$ cannot be diminished; for the smallest circle which will cover an equilateral triangle of side 1 is its circumcircle, and the radius of this circle is $1/\sqrt{3}$. The problem of the circular lid is therefore completely solved, and we see that the radius of the smallest circular lid is $1/\sqrt{3}$.

Naturally, one can consider the analogous problem for lids of shapes other than the circular. It is, for example, almost trivial that the smallest square which will cover any figure of diameter 1 has side 1. To my knowledge only two other lid problems have been solved; thus it is known, and not very difficult to prove, that the smallest equilateral triangle which will cover any figure of diameter 1 has side $\sqrt{3}$, and that the smallest regular hexagon with the same property has side $1/\sqrt{3}$. A much harder problem arises if we leave unspecified the shape of the lid and merely require that its area should be as small as possible. In other words, we ask: what is the figure of least area which will cover any figure of diameter 1? This question was proposed more than forty years ago, but the answer is still unknown except that the figure we look for must be very irregular indeed.



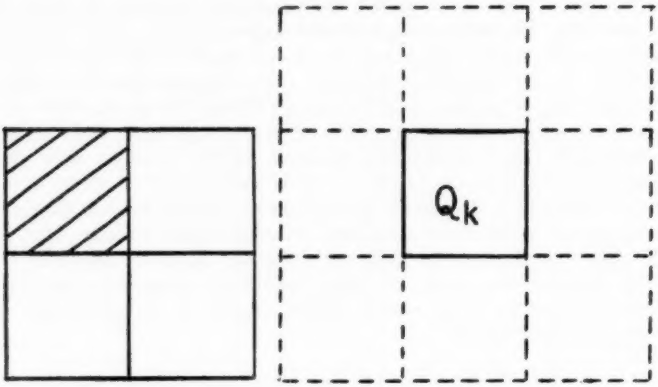
The overlap problem

In the lid problem the object was to cover one figure by another. The next question we discuss is also concerned with covering, but this time with *partial* covering.

Consider a finite system S of squares, not necessarily of equal size, with their sides parallel to two fixed (perpendicular) lines. In general the squares of S will overlap to some extent, and the system will cover a certain area which we shall denote by $|S|$. Our problem is to investigate what proportion of this area can be covered by a *non-overlapping*[1] subsystem of S. In other words, given a system S, how large can we make the ratio $|S_0|/|S|$ by a suitable choice of a non-overlapping subsystem S_0? One result is obvious at once; in some cases this ratio cannot become greater than $\frac{1}{4}$. For let S consist of four squares of equal size making up a larger square; then S_0 can only be chosen as one of the original four squares[2], and we therefore

[1] A system of figures is said to be non-overlapping if no two figures have a point in common.

[2] It is assumed that the squares we consider are closed: this convention does not affect the nature of the problem.



have $|S_0| = \frac{1}{4}|S|$. Thus, in *some* cases, we can only cover one quarter of the original area. What is much more interesting is that, in *every* case, we can cover at least a *fixed* proportion of the original area. Indeed, we shall show that, given any system S, we can always select a non-overlapping subsystem S_0 such that

$$|S_0| \geqslant \tfrac{1}{9}|S|. \tag{4}$$

The proof of (4) is quite easy. In the course of it we shall make use of the obvious inequality $|S| + |T| \geqslant |S + T|$; here S, T are arbitrary systems of squares and $S + T$ denotes the system of all squares either in S or in T. An analogous inequality holds, of course, for more than two systems.

Write $S = S_1$, and denote by Q_1 the largest square in S_1[1], and by T_1 the subsystem of S_1 consisting of those squares (including Q_1) which have at least one point in common with Q_1. Let S_2 be the system obtained when the squares of T_1 have been removed from S_1. Again, let Q_2 be the largest square in S_2, and denote by T_2 the subsystem of S_2 consisting of those squares (including Q_2) which have at least one point in common with Q_2. Denote by S_3 the system obtained when the squares of T_2 have been removed from S_2, and let Q_3 be the largest square in S_3. Continuing in this manner as long as there are any squares left, we obtain a non-overlapping subsystem $S_0 = \{Q_1, Q_2, Q_3, \ldots\}$ of S. It is, moreover, clear that

$$S = T_1 + T_2 + T_3 + \ldots .$$

Now Q_k is the largest square in S_k. Hence any square in S_k which has a point in common with Q_k must be contained within a square of

[1] If S_1 contains several squares of maximal size, we select any one of them.

area $9|Q_k|$. Thus $|T_k| \leqslant 9|Q_k|$ $(k = 1, 2, 3, \ldots)$ and therefore

$$\begin{aligned}|S_0| &= |Q_1| + |Q_2| + |Q_3| + \ldots \\ &\geqslant \tfrac{1}{9}\{|T_1| + |T_2| + |T_3| + \ldots\} \\ &\geqslant \tfrac{1}{9}|T_1 + T_2 + T_3 + \ldots| \\ &= \tfrac{1}{9}|S|,\end{aligned}$$

as asserted. It is worth observing that our argument has given us more than the required inequality, for we have also described a definite procedure for selecting a non-overlapping subsystem S_0 satisfying (4).

To carry the discussion further, it is useful to introduce the notion of a 'covering constant.' Let $\mathfrak{C}$ be a class of systems of plane figures (e.g. the class of systems of similarly situated squares of arbitrary size.) Denote by $\sigma = \sigma(\mathfrak{C})$ the largest number with the property that, for any system S of class $\mathfrak{C}$, we can find a non-overlapping subsystem S_0 such that $|S_0| \geqslant \sigma\,|S|$. We shall call σ the *covering constant* of $\mathfrak{C}$. The general overlap problem consists in determining σ for any given class $\mathfrak{C}$.

So far we have shown that, for systems of similarly situated squares of arbitrary size,

$$\tfrac{1}{4} \geqslant \sigma \geqslant \tfrac{1}{9}.$$

It is not altogether easy to narrow these bounds, and only a slight improvement has been achieved up to the present. Thus it is known that $\sigma > \frac{4}{35}$, while it is conjectured that

$$\sigma = \tfrac{1}{4}. \tag{5}$$

It seems very difficult to make further progress in this problem. However, if we confine ourselves to systems of similarly situated squares of *equal* size, the situation becomes much more tractable and in this case (5) can be proved. Again, for systems of congruent and similarly situated triangles, $\sigma = \frac{1}{6}$; while for systems of regular, similarly situated hexagons of equal size we have $\sigma = \frac{1}{4}$. For systems of equal circles the problem has not yet been solved completely; in this case the best known bounds for σ are given by the inequalities

$$\tfrac{1}{4} \geqslant \sigma \geqslant \frac{\pi}{8\sqrt{3}} = 0{\cdot}226\ldots.$$

The turning problem

The last question I propose to consider is concerned with a very special class of figures which I shall call *T-figures*. A T-figure is a figure having the property that is is possible to move a segment of length 1 continuously inside it in such a way that it turns through 360° and ultimately returns to its original position. An obvious

instance of a T-figure is a circle of diameter 1; but, as we shall see, this is by no means the only instance. We shall now discuss briefly the problem of determining the T-figure of least area. This problem will be referred to as the turning problem.

The nature of the turning problem depends of course, on the choice of figures which are to be regarded as eligible for the competition. If we restrict our choice to convex figures,[1] the problem is not unreasonably difficult and it can be shown that the equilateral triangle of side $2/\sqrt{3}$ (and area $1/\sqrt{3}$) is the convex T-figure of least area. (The fact that this triangle is a T-figure is obvious since its altitudes are of length 1.)

If the requirement of convexity is abandoned, the question becomes much harder and also much more interesting. Let us begin by constructing an example of a non-convex T-figure. Consider a circle of radius $\frac{1}{4}$ rolling inside a fixed circle of radius $\frac{3}{4}$; a point on the circumference of the rolling circle then traces out a curve known as the *three-cusped hypocycloid*. This curve has the striking property that it intercepts a segment of constant length 1 on any tangent drawn to it. From this we infer at once that the figure bounded by the hypocycloid is a T-figure. Its area is $\frac{1}{8}\pi$, a number appreciably smaller than $1/\sqrt{3}$. Hence, if convexity is not demanded, we can practise much greater economy in the construction of T-figures.

It was believed for some time that the hypocycloid just described is, in fact, the T-figure of least area. This conjecture, though extremely plausible, is nevertheless false; and the correct answer turns out to be very surprising. Briefly, the solution of the problem is that there is no solution; that is, there exists no T-figure of least area. To be more explicit: given any positive number ε, no matter how small, we can find a T-figure whose area is less than ε. This is, indeed, a barely credible conclusion and it is difficult to grasp intuitively the fact that a segment can be turned round completely inside a figure of arbitrarily small area. Yet it need hardly be stressed that the construction of a T-figure of small area involves a very complicated procedure and that the resulting figure is not at all easy to visualize.

With the unexpected twist in the turning problem I conclude my survey. Problems of the type I have discussed are admittedly not of the highest 'importance' and they will not influence the general trend of mathematical research in the way it has been influenced, for example, by the introduction of coordinates into geometry or Lebesgue integration into analysis. But though these problems are to be found in the byways rather than the highways of mathematics, they

[1] A figure is said to be *convex* if, whenever the points A and B belong to it, so does the entire segment AB.

cannot fail to exert a peculiar fascination. Most of them are wolves in sheep's clothing, for the deceptive air of simplicity often conceals an unexpected toughness. For this reason, if for no other, they will continue to offer a challenge to the ingenuity of mathematicians.

Bibliographical note

A problem closely related to the network problem was discussed by L. Fejes Tóth, *Math. Zeitschrift* 46 (1940), 83–85 and subsequently by S. Verblunsky, *Proc. Amer. Math. Soc.* 2 (1951), 904–913. The method indicated above and the inequality (3) were found by L. Few, *Mathematika* 2 (1955), 141–144.

The problem of the circular lid is classical. It was mooted in the writings of J. J. Sylvester and G. Chrystal; but in its present form was proposed and solved by H. W. E. Jung, *J. für Math.* 123 (1901), 241–257 and 137 (1910), 310–313. Jung also dealt with the analogous problem in n-dimensional space, but his treatment is complicated and much simpler solutions have been given since; see e.g. L. M. Blumenthal and G. E. Wahlin, *Bull. Amer. Math. Soc.* 47 (1941), 771–777. The theorem on the triangular lid is a special case of a result due to D. Gale, *Proc. Amer. Math. Soc.* 4 (1953), 222–225; the theorem on the hexagonal lid was proved by J. Pál, *Math.-fys. Medd. Danske Vid. Selsk.* 3(1920), 35 pp. The problem concerning the lid of unspecified shape and minimal area was proposed by H. Lebesgue, *Bull. Soc. Math. France C.R.* (1914), 72–76. Lebesgue was also the first to use the term 'lid' (*couvercle*).

J. C. Burkill, *Proc. Cambridge Phil. Soc.* 21 (1923), 659–663 noted the inequality (4) and, indeed, the corresponding inequality in n dimensions. All other results on the overlap problem discussed above are due to R. Rado, *Proc. London Math. Soc.* (2), 51 (1949), 232–264. For a series of closely related questions see Rado, *Proc. London Math. Soc.* (2) 53 (1951), 243–267.

What we have called the 'turning problem' is usually referred to as Kakeya's problem having been raised by S. Kakeya, *Sci. Reports Tôhoku Univ.* 6 (1917), 71–88. It was solved for convex figures by J. Pál, *Math. Annalen* 83 (1921), 311–319, and for arbitrary figures by A. S. Besicovitch, *Math. Zeitschrift* 27 (1927), 312–320. Besicovitch's solution was simplified appreciably by O. Perron, *Math. Zeitschrift* 28 (1928), 383–386.

University of Sheffield L. M.

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

Dear Sir,

May I add shortly to what Mr. Snell has written in the Gazette about our mutual friend Arthur Siddons?

I would like to emphasize how helpful he was to young schoolmasters, and never seemed to grudge the time involved. There is no one to whom I owe a greater debt in learning how to teach Mathematics. We had much in common. He succeeded my father at Harrow. We had taken the same two Triposes at Cambridge, though my result was far less distinguished. I often went to see him during School holidays to put forward difficulties which he soon resolved. At a later date, about 1913, the school I was then serving provided a Mathematical laboratory. The work I did there was entirely based on that done in his pioneer laboratory at Harrow.

He had ambitions in early days to become a headmaster. I often told him, and still believe, that this would have interfered with his work for Mathematical reform. Other interests would inevitably have come his way, and I did not regret that he remained where he was.

I would like to stress that our Association owes him an incalculable debt for his pioneer work some sixty years ago. Prejudice had to be overcome. Committees had to be persuaded. Time and patience were necessary. No doubt the reform of Mathematical teaching was due to many, but his contribution was notable, and he stands in the forefront among those whose services we remember with gratitude.

Yours etc.,

W. F. Bushell

To the Editor of the *Mathematical Gazette*

Dear Sir,

In the review of Ministry of Education Pamphlet 36 in last February's Gazette the introduction of certain new topics is approved, including "some of the newer branches of mathematics, e.g. symbolic logic and Boolean algebra". We may well feel that they should have a place with us, when, as has been said, their inventor, George Boole, in discovering them discovered Pure Mathematics. Who was Boole? He lived and taught in England and Ireland between 1815 and 1864. For eighteen years he was a school teacher, starting as an 'usher' at the age of 16, later on having a school of his own. According to E. T. Bell, it was during this latter period that impetus was given to his work in mathematics by his dismay at the then available text books. All his own higher education he got by part-time study; and then in his spare time he started his great original work with which he "made a far-reaching advance in mathematical methods". Mr. Flemming, in his review, suggested that we should be keeping abreast of the times and helping to close the gap between school and University. And so indeed we should. But the

matter has I think a still wider significance and importance: it is put in the Pamphlet that "a lesson or two on Boolean Algebra (or another 'modern' algebra) with its novel operational rules might reveal more clearly, by contrast, what 'ordinary' Algebra is and does, and also give a taste of symbolic logic to those whose appetites and digestions are suited to it." The suggestion being that, even without taking Boolean Algebra a long way for its own sake but remaining content with an introduction, we shall be giving valuable experience which cannot fail to enhance understanding of what algebra is and does. I think this will be found to be the case.

Is there any evidence to the point? None that I know. It seems to me important that we should collect some: for this sort of increase of understanding is surely one of our primary aims. I should be extremely interested to hear from anyone who would help to collect evidence on this topic. When? and How? Your syllabus may allow you a slight relaxation, after examinations perhaps or at any rate toward the end of the Summer term—a new topic at these times is often a relief and a refreshment. I need hardly say that I do not propose we take Mr. Hooley's article on Sentence Logic as a text (though of course this logic is a Boolean algebra); rather I suggest an introductory course that might be a father to his precocious child, developing an algebra from immediate or commonsense notions in a way applicable at any level in a secondary school. Since no text exists for a naïve introduction, my interest has led me to prepare very full annotated lesson notes adaptable for any level. To those readers who are interested I will gladly send duplicated copies of my notes to try and test what there is in the idea.

Requests to me at 56 Vicars Hill, S.E.13.

Yours sincerely,

PETER CALDWELL

To the Editor of the *Mathematical Gazette*

Dear Sir,

S. Inman suggests that the phrase "Take away" is an artificiality which should be abolished. Why? What's artificial about it? To say that subtraction is simply being given the sum of two numbers and one of them and being asked to find the other, is merely one way of looking at the question. It is certainly not the only way.

I quote from his letter: "Of course, I am describing the method very briefly and I am not dealing with the gradual build-up which is needed for young children." Quite! It would be interesting (to me) to know how the build-up would proceed, using, say, bundles of sticks, which is what infants employ in the early stages.

The method mentioned by Mr. Inman is very good "on paper", but I suggest the practical demonstration of it is not going to be quite so easy. Credit is, indeed, due to Miss Burslem for her attempt to grapple with the problem. But to ask her to scrap what has been found to be successful in practice and start again "on the lines which I have indicated" strikes me as being just a little bit . . . . Has Mr. Inman ever taught infants? . . . A very relevant question, believe me.

Finally, it may be very naughty of me, but I confess to being unimpressed by the battery of famous names (including as it does, four past Presidents of the Association). Dare I ask also: Has any of these gentlemen ever taught Infants?

Yours etc.
R. V. Parker

To the Editor of the *Mathematical Gazette*

Dear Sir,

I like the method of beginning subtraction suggested by Miss Burslem in your October number, 1959, but I cannot agree that equal additions is as hard to demonstrate as she suggests. Equal additions is the method of subtraction that derives from problems in which the difference must be found. If a child has to find the difference between a 6 inch rod and a 9-inch rod, he has both minuend and subtrahend before him, and both do have an existence. It is then, also, quite easy to show that equal additions do not alter the difference. Children see this quickly when their own ages are used in examples.

This method of subtraction has some advantages, one of which includes the appreciation of the basic idea that equal additions do not alter differences, but this is appropriate to a year or more of mental growth beyond decompositions, and should probably not be attempted until then.

The difficulty with crutches is easily dealt with by always adding one to the bottom at the same time as one to the top line, where that is necessary.

Many teachers object to teaching more than one process of subtraction in the scare of confusion. I doubt if this is reasonable for I have observed that, without knowing it, teachers who use equal additions for whole numbers will use decomposition for fractions—and without confusing the children.

Yours sincerely,
Ray Chapman-Taylor

WANTED

Mathematical Questions from the Educational Times, in particular the volumes by W. I. C. Miller.

Robert Simson's Euklid from 1756, and those of I. Playfair, Ch. Hutton, T. Perronet Thompson, A. de Morgan, Todhunter, R. Townsend, Nixon. I. MacMahon, Elementary Plane Geometry, 1903.

Offers to Karl Michaelis,
Hamburg-Wandsbek,
Rodigallee 100.

55. A problem on friction

The following question was set in the Statics and Dynamics paper of the Oxford and Cambridge Mathematics for Science (Advanced Level) examination in June 1951:

A bead A, of weight W, is threaded on a rough circular wire, whose centre is O and whose plane is vertical; the bead is attached to a light inextensible string which passes through a fixed smooth hook B at the upper end of the vertical diameter and carries a weight W_1; the angle BOA is denoted by 2θ. When the bead is on the point of slipping downward, $\theta = \alpha$; when it is on the point of slipping upward $\theta = \beta$. Prove that, if $\alpha + \beta = \dfrac{\pi}{2}$, the coefficient of friction between the bead and the wire must be $\tan(\alpha - \frac{1}{4}\pi)$. Prove also that

$$W_1 = W(\sin\alpha + \cos\alpha).$$

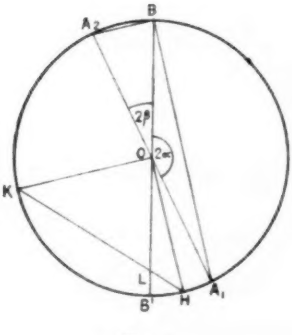


Fig. 1

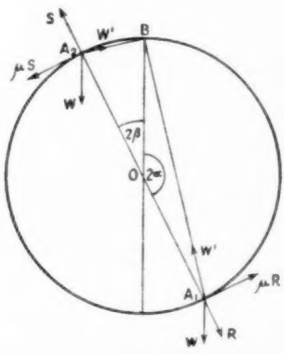


Fig. 2

The required results are easily obtained, either from the composite figure 1, in which OH, OK are parallel to BA_1, BA_2 and triangles OLH and OLK are taken as triangles of forces, or from figure 2 by resolving along the tangent and radius at A_1 and A_2, *provided we assume that the normal reaction R acts outwards*. The question arises, —can R act inwards? Can we, in fig. 1, find a point L' on BB' such that HL' and KL' are equally inclined to A_1A_2 but on opposite sides?

Now the locus of a point P such that HP and KP are equally inclined to A_1A_2 but on opposite sides is very easily shown by coordinate geometry to be a rectangular hyperbola through H and K, whose centre bisects HK and whose asymptotes are parallel and

perpendicular to A_1A_2. It would appear likely that in favourable circumstances this hyperbola might cut BB' in two points, each a possible position of L'. (See fig. 3)

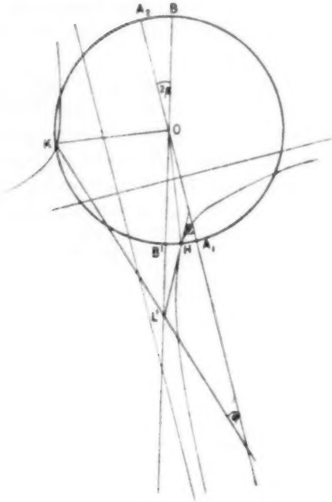


Fig. 3

If the equations of resolution are revised for the change in sense of R, we obtain, after elimination,

$$X^2 \cos 2\beta - X(\sin 3\beta + \cos 3\beta) + \sin 4\beta = 0$$

$$\text{where } X = \frac{W'}{W}$$

and this equation gives real values for X if $4s^3 - 5s + 1 \geqslant 0$ where $s = \sin 2\beta$. Now $4s^3 - 5s + 1 = 0$ when $s = 1$, $0{\cdot}207$ or $-1{\cdot}2$ approx., and hence we obtain real values of X if $0 < \sin 2\beta \leq 0{\cdot}2$. Taking as an example $\beta = 4°$, or $\sin 2\beta \doteqdot 0{\cdot}14$, we get $X \doteqdot 0{\cdot}88$ or $0{\cdot}32$; i.e., $W' \doteqdot \frac{8}{9}W$ or $\frac{1}{3}W$.

This means that, in fig. 3, if β were $4°$, the hyperbola would cut BOB' in two points, say L_1' and L_2' which make $\theta = \phi$ and such that $HO \doteqdot \frac{8}{9}OL_1'$ and $HO \doteqdot \frac{1}{3}OL_2'$.

Thus if β is less than about $6°$ there are, in addition to the expected solution, two solutions in which the reaction on the bead in the lower position acts inwards, and in which the required results do not hold.

City of London School R. E. Green

56. On class room note 39

The interesting property of cubic graphs in Note 39 can be proved without the use of calculus as follows:

If the cubic is given by $y = k(x - a)(x - b)(x - c)$, and any line through $(c, 0)$ is taken to be $y = m(x - c)$, where m can vary, then the line meets the curve where

$$k(x - a)(x - b)(x - c) = m(x - c)$$

$\therefore$ apart from $x = c$, the x-values of the points of intersection are given by

$$x^2 - (a + b)x = \frac{m}{k} - ab$$

These values coincide if

$$x = \frac{a + b}{2} \quad \text{and} \quad -\frac{(a + b)^2}{4} = \frac{m}{k} - ab$$

i.e.

$$m = -\frac{k}{4}(a - b)^2$$

Thus the tangent at $x = \dfrac{a + b}{2}$ passes through $(c, 0)$ and has gradient $-\dfrac{k}{4}(a - b)^2$.

Eton College D. G. BOUSFIELD

Editorial note. A similar solution was received from A. R. Pargeter.

57. On class room note 39

Mr. Nakazawa's interesting result may also be proved in this way.

By shifting the origin a suitable distance along the x axis, the equation of the cubic curve may be expressed in the form

$$y = Ax^3 + Bx + C. \tag{1}$$

Lemma

Any line (not parallel to the y axis) meets the curve in three points (two of which may be imaginary) for which the sum of the x coordinates is zero, as we see by substituting $y = mx + n$.

From the lemma, we see that for the three points where the curve meets the x axis, we have $a + b + c = 0$. Using the lemma again, we see that the tangent at the point for which $x = \frac{1}{2}(a + b) = -\frac{1}{2}c$ meets the curve again at the point for which $x = c$.

Since the proof is independent of C, the result is true for any line parallel to the x axis.

Incidentally, we can also give an alternative proof of Mr. Hurrell's result in Note 2635. If we start from equation (1), a shift of the origin along the y axis enables us to express the equation in the form

$$y = Ax^3 + Bx.$$

The curve is symmetrical about the new origin, and so are the two stationary points.

University of Leicester E. J. F. PRIMROSE

58. Invariants of a conic

In the Association's report *The Teaching of Higher Geometry in Schools*, the invariance of $a + b$ and $ab - h^2$ is proved by considering a transformation of

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2)$$

and the invariance of Δ is proved by considering a transformation of $S + \lambda$ (section 8.6). This seems rather artificial: why choose just these particular expressions?

The invariants may be derived naturally from the process for finding the axes of the conic (section 8.43). A change of origin reduces the general equation to

$$ax^2 + 2hxy + by^2 + k = 0,$$

where $k = \Delta/(ab - h^2)$, that is,

$$a_1x^2 + 2h_1xy + b_1y^2 = 1,$$

where $a_1 = -a/k$, $h_1 = -h/k$, $b_1 = -b/k$. We now show, as in 8.43, that if the equation of the conic referred to its principal axes is

$$\alpha\xi^2 + \beta\eta^2 = 1,$$

then α and β are the roots of

$$\lambda^2 - \lambda(a_1 + b_1) + a_1b_1 - h_1{}^2 = 0,$$

that is,

$$\lambda^2 + \lambda(a + b)/k + (ab - h^2)/k^2 = 0.$$

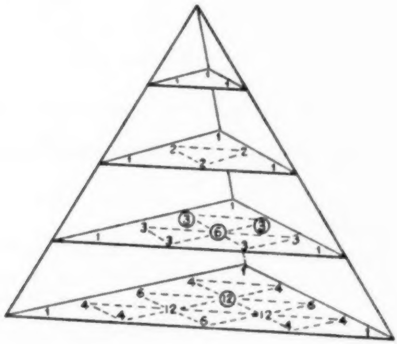
Since α and β are intrinsic numbers associated with the conic, it follows that $(a + b)/k$ and $(ab - h^2)/k^2$, that is, $(a + b)(ab - h^2)/\Delta$ and $(ab - h^2)^3/\Delta^2$, are absolute invariants of the conic.

It is true that this method does not show that $a + b$, $ab - h^2$ and Δ are relative invariants, but it is the absolute invariants which are really important.

E. J. F. PRIMROSE

2901. The Pascal tetrahedron

Last July, at the end of his fourth year in a grammar school, John Strange was introduced to Pascal's Triangle during some work on the binomial expansion. The idea of extending its application immediately took hold of him, and he has since evolved the following method of calculating the coefficients in a trinomial expansion, say $(a+b+c)^4$.



The diagram represents the model he constructed of equilateral cardboard triangles joined at the vertices by strips of balsa wood to form a regular tetrahedron. Each number is the sum of its three nearest neighbours in the layer above: for instance, the ringed 12 is the sum of the ringed 6 and 3's. The numbers in the nth layer give the coefficients in the expansion of $(a+b+c)^n$, allocated according to the scheme

$$\begin{array}{c}
a^n \\
a^{n-1}b \quad a^{n-1}c \\
a^{n-2}b^2 \quad a^{n-2}bc \quad a^{n-2}c^2 \\
a^{n-3}b^3 \quad a^{n-3}b^2c \quad a^{n-3}bc^2 \quad a^{n-3}c^3 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
b^n \quad b^{n-1}c \quad b^{n-2}c^2 \quad \cdots\cdots \quad b^2c^{n-2} \quad bc^{n-1} \quad c^n
\end{array}$$

The method depends of course, on the identity

$$\frac{n!}{p!q!r!} = \frac{(n-1)!}{(p-1)!q!r!} + \frac{(n-1)!}{p!(q-1)!r!} + \frac{(n-1)!}{p!q!(r-1)!}$$

where $p+q+r=n$.

Minchenden School J. L. Atkin

2902. A family of equilateral polyhedra

A series of polyhedra with 2 n-gonal faces and $2n$ pentagonal faces is mentioned by Wells [*The Third Dimension in Chemistry, p.* 40]. This note establishes conditions which must be satisfied by such polyhedra if they have the symmetry of n-gonal prismoids and the pentagons are equilateral.

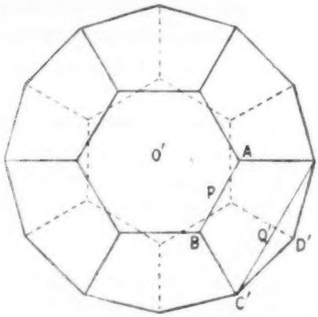


Fig. 1

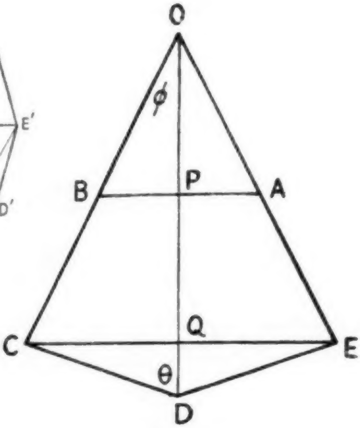


Fig 2.

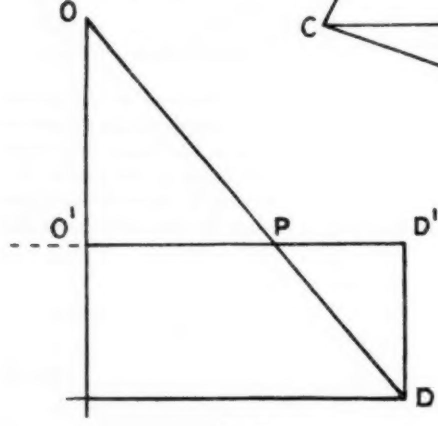


Fig. 3

Fig. 1 is a projection of one of the hexagonal members of the family onto a plane parallel to a hexagonal face.

Fig. 2 is in the plane of a pentagonal face ABCDE and the conditions are worked out in terms of the angles θ and ϕ in this figure. e is the edge-length.

Fig. 3 is in the plane OPQ, perpendicular to Fig. 1.

From Fig. 2 $$PQ = OQ - OP,$$

i.e. $$e\cos\phi = e\cot\phi(\sin\theta - \tfrac{1}{2})$$
which is best written
$$\sin\phi = \sin\theta - \tfrac{1}{2} \tag{1}$$

Since ABCDE is a plane pentagon, we have from Fig. 3

$$\frac{OP}{O'P} = \frac{PD}{PD'}$$

i.e. $$\frac{\tfrac{1}{2}e\cot\phi}{\tfrac{1}{2}e\cot\dfrac{\pi}{n}} = \frac{e(\cos\phi + \cos\theta)}{e\sin\theta\operatorname{cosec}\dfrac{\pi}{n} - \tfrac{1}{2}e\cot\dfrac{\pi}{n}}$$

Eliminating ϕ between this expression and (1), and writing $x = \sin\theta$ and $c = \cos\dfrac{\pi}{n}$, we obtain the following quartic equation for x:

$$(x + 1)(x - \tfrac{1}{2})^2(x - 1) - (x - \tfrac{1}{4})\{(1 - c)^2/(1 - 2c)\} = 0 \tag{2}$$

Before solving this equation, one more condition must be noted. The angles of a pentagonal face are as follows:

$$2\theta, \quad \pi - \theta - \phi \text{ (2 angles)}, \quad \frac{\pi}{2} + \phi \text{ (2 angles)}.$$

For a convex polyhedron to be possible the sum of the face angles at any one point is less than 2π.
So at a vertex like A,

$$2\left(\frac{\pi}{2} + \phi\right) + \left(\pi - \frac{2\pi}{n}\right) < 2\pi$$

i.e. $$\phi < \frac{\pi}{n}$$

or $$\sin\theta < \tfrac{1}{2} + \sin\frac{\pi}{n}. \tag{3}$$

Because of (1), all relevant solutions of (2) lie in the range $\tfrac{1}{2} < x < 1$. A graph was drawn of

$$y = (x + 1)(x - \tfrac{1}{2})^2(x - 1)$$
and
$$y = (x - \tfrac{1}{4})\{(1 - c)^2/(1 - 2c)\} \quad \text{for} \quad n = 5, 6, \ldots 10.$$

Two coincident solutions were found for $n = 5$, as might be expected and can be checked by putting $x = c = \tfrac{1}{4}(1 + \sqrt{5})$ in (2) and its derivative. The larger solution for $n \geqslant 7$ was discarded because (3) was not satisfied.

TABLE OF DATA

				Angles of Pentagon			Dihedral Angles			
n	$\sin\theta$	θ	ϕ	2θ	$\pi-(\theta+\phi)$	$\frac{\pi}{2}+\theta$	(χ) $n-5$	(a) $5-5$	(b) $5-5$	$\frac{h}{2R}$
6 (i)	0·9579	73° 19′	27° 15′	146° 38′	79° 26′	117° 15′	153°	153° 54′	49° 43′	0·245
5	0·8090	54°	18°	108°	108°	108°	116° 34′	116° 34′	116° 34′	0·809
6 (ii)	0·6212	38° 26′	6° 59′	76° 52′	134° 35′	96° 59′	102° 16′	121° 30′	140° 37′	1·088
7	0·5776	35° 17′	4° 27′	70° 34′	140° 16′	94° 27′	99° 18′	129° 18′	148° 21′	1·042
8	0·5549	33° 42′	3° 9′	67° 24′	143° 9′	93° 9′	97° 37′	135° 26′	153° 3′	0·967
9	0·5413	32° 46′	2° 22′	65° 32′	144° 52′	92° 22′	96° 31′	140° 16′	156° 10′	0·891
10	0·5323	32° 10′	1° 51′	64° 20′	145° 59′	91° 51′	95° 43′	144° 11′	158° 42′	0·822
$\to\infty$	$\to 0{\cdot}5$	$\to 30°$	$\to 0°$	$\to 60°$	$\to 150°$	$\to 90°$	$\to 90°$	$\to 180°$	$\to 180°$	$\to 0$

(a) The dihedral angle between two faces adjacent to the same n-gon. (b) The dihedral angle between two faces adjacent to opposite n-gons. h is the distance between parallel n-gons and R the maximum radius parallel to the n-gons. If e is length of an edge.

$$\frac{h}{e} = (2\cos\phi + \cos\theta)\sin\chi; \quad \frac{R}{e} = \sin\theta \operatorname{cosec}\frac{\pi}{n}$$

We are left with the following polyhedra

1. The regular dodecahedron.
2. Two different hexagonal members of the family.
3. A series of polyhedra, one for each $n \geqslant 7$.

Approximate solutions from the graph were adjusted by applying Newton's Approximation twice and the table of values above obtained.

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2903. On triangular numbers

1. Let T_n denote the nth triangular number $\frac{1}{2}n(n+1)$, then

$$T_a + T_b + ab = T_{a+b};$$

for

$$T_{a+b} = \frac{(a+b)(a+b+1)}{2}$$

$$= \frac{(a+b)^2 + (a+b)}{2}$$

$$= \frac{a(a+1)}{2} + \frac{b(b+1)}{2} + ab$$

$$= T_a + T_b + ab.$$

Hence, taking $a = 5$, $b = 21$ so that $ab = 5 \,.\, 21 = \frac{1}{2} \,.\, 14 \,.\, 15$,

$$T_5 + T_{21} + T_{14} = T_{26}$$

and similarly

$$T_6 + T_{20} + T_{15} = T_{26}$$

$$T_{20} + T_{77} + T_{55} = T_{97}$$

$$T_{21} + T_{76} + T_{56} = T_{97}$$

and so on

2. The three triangular numbers T_2, T_4, T_5 have the following properties

2.1 The product of any two added to the third is a triangular number

2.2 The sum of the three is a triangular number

2.3 The sum of the products two at a time is a triangular number.

2.4 The product of the three is twice the sum of the products two at a time

For $T_2T_4 + T_5 = T_9$, $T_2T_5 + T_4 = T_{10}$, $T_4T_5 + T_2 = T_{17}$,

$$T_2 + T_4 + T_5 = T_7$$

$$T_2T_4 + T_2T_5 + T_4T_5 = 15^2$$

and $$T_2T_4T_5 = 2 \times 15^2.$$

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2904. A curiosity

We show that integers a, b, c, d, satisfying

$$a + b = c + d$$

may be found to make

$$ab + cd$$

a square.

Take $c = a + 1$, $d = b - 1$ and

$$ab + cd = (b - c)^2$$

so that

$$(a + b)^2 - 3(a - b + 1)^2 = 1.$$

Solutions of the Pell's equation

$$x^2 - 3y^2 = 1$$

are

$$x = 2,\ 7,\ 26,\ 97,\ \ldots$$
$$y = 1,\ 4,\ 15,\ 56,\ \ldots$$

giving

$$a = 1,\ 5,\ 20,\ 76,\ \ldots$$
$$b = 1,\ 2,\ 6,\ 21,\ \ldots$$

the corresponding values of c, d being given by the conditions $c = a + 1$, $d = b - 1$.

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2905. A generalized arithmetico-geometric series

The series

$$\frac{x^k}{r} + \frac{(x+d)^k}{r^2} + \frac{(x+2d)^k}{r^3} + \frac{(x+3d)^k}{r^4} + \ldots$$

is absolutely convergent for all r satisfying $|r| > 1$; if $g_k(x)$ is its sum then

$$g_k(x) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{s=0}^{k} \binom{k}{s} x^{k-s} d^s n^s$$

$$= \sum_{s=0}^{k} \binom{k}{s} x^{k-s} d^s \sum_{n=0}^{\infty} \frac{n^s}{r^{n+1}}.$$

Let $$\beta_s = \sum_{n=0}^{\infty} \frac{n^s}{r^{n+1}}, \quad \text{then} \quad g_k(x) = \sum_{s=0}^{k} \binom{k}{s} x^{k-s} d^s \beta_s;$$

but $$g_k(d) = rg_k(0)$$
and so
$$d^k \sum_{s=0}^{k} \binom{k}{s} \beta_s = r\beta_k d^k$$
Whence
$$(r-1)\beta_k = \sum_{s=0}^{k-1} \binom{k}{s}\beta_s, \text{ for all } k,$$
which together with the initial condition
$$\beta_0 = \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \ldots = \frac{1}{r-1}$$
determines β_k for all k.
Alternatively, writing $y_s = (r-1)^{s+1}\beta_s$ so that
$$y_0 = 1, \quad y_k = \sum_{s=0}^{k-1} (r-1)^{k-s-1}\binom{k}{s} y_s, \quad \text{for all } k,$$
we have
$$g_k(x) = \sum_{s=0}^{k} \binom{k}{s} \frac{x^{k-s} d^s y_s}{(r-1)^{s+1}}.$$

The following results, due to Ramanujan, are immediate consequences of this formula:

(i) $\dfrac{1^5}{2} + \dfrac{2^5}{2^2} + \dfrac{3^5}{2^3} + \ldots = 1082$

(ii) $\dfrac{1^5}{3} + \dfrac{2^5}{3^2} + \dfrac{3^5}{3^3} + \ldots = 68\frac{1}{4}$

Another simple consequence is that, for $n \geq 2$,
$$\sum_{m=1}^{\infty} \frac{m^n}{n^m} = \frac{n\delta_n}{(n-1)^{n+1}},$$
where $\delta_0 = 1$,
$$\delta_s = \sum_{t=1}^{s} (n-1)^{s-t} \binom{s}{t-1} \delta_{t-1}$$

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2906. A maximum property of two concentric circles

β, γ are two concentric circles with β inside γ, and P is a point inside β. A half line through P meets β and γ at B, C respectively. Then BC is greatest when PBC is at right angles to the diameter through P. It is of course very easy to prove this using the Calculus; the object of this note is to prove the result by pure geometry.

Let O be the common centre of β, γ and let PB_0C_0 be a half line through P, perpendicular to the diameter PO, meeting β, γ at B_0, C_0 respectively, and let PB_LC_L be any other half line through P meeting β, γ at B_L, C_L Further let a circle, centre O, touch PB_LC_L at A, and cut PO at A', and let $A'B'C'$ be the tangent to this circle

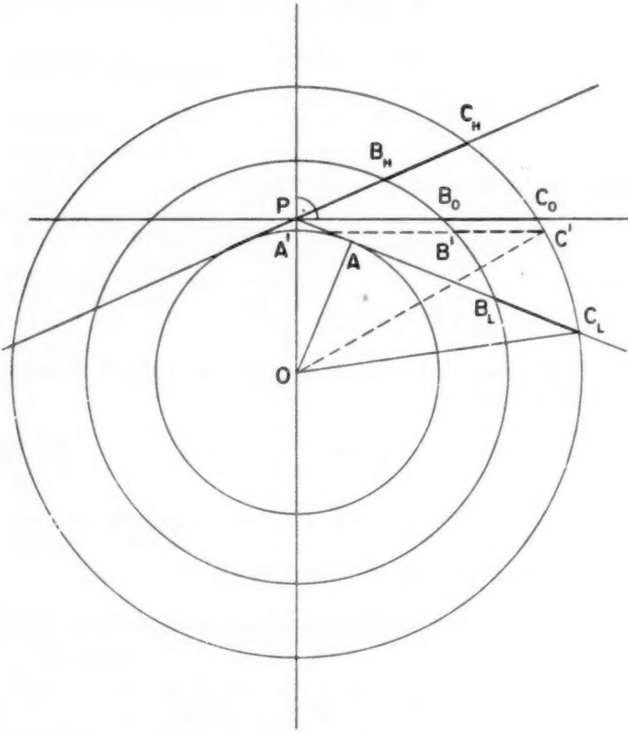

FIG. 1

at A', meeting β, γ at B', C' (Fig. 1). Then triangles OAC_L, $OA'C'$ are congruent, and so are triangles OAB_L, $OA'B'$ and therefore $B_LC_L = B'C'$. It remains to prove that $B_0C_0 > B'C'$.

Let two parallel lines meet β, γ at B_1, C_1 and B_2, C_2 (Fig. 2) and let P, Q be the mid points of B_1B_2, C_1C_2 (so that PQ is parallel to B_1C_1) Then, if B_1C_1 is further from O than B_2C_2, we have $B_1C_1 > B_2C_2$.

Let OP, OQ make angle θ, θ' with B_2C_2 (so that $\theta > \theta'$) and let B_1B_2, C_1C_2 make angles ϕ, ϕ' with B_2C_2 so that $\theta + \phi = \theta' + \phi' =$

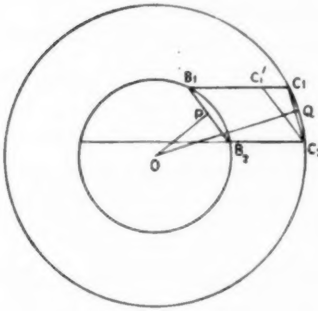

Fig. 2

$90°$, and therefore $\phi' > \phi$. Draw C_2C_1' parallel to B_1B_2 meeting B_1C_1 at C_1'; since $\phi' > \phi$, therefore C_1' lies between B_1 and C_1, and so $B_2C_2 = B_1C_1' < B_1C_1$, which completes the proof

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2907. On Note 2838

The following elementary approach yields continuous solutions to a wide class of functional equations:

$$af(x) = f(x + b)$$

Suppose $f(x)$ is continuous and equal to A at $x = B$. Then:

$$aA = f(B + b)$$

$$a^2A = f(B + 2b)$$

. . . .

$$a^nA = f(B + nb)$$

Eliminating n we have only to write $x = B + nb$, whence:

$$f(x) = a^{\frac{x-B}{b}} A$$

or, without loss of generality

$$f(x) = Ca^{\frac{x}{b}}$$

E. DE ST. Q. ISAACSON

2908. Solution of Problem 2838

We shall determine (a) all solutions, (b) all continuous solutions of the functional equation

$$af(x) = f(x + b), \tag{1}$$

where a and b are constants.

First of all we note that evidently *for* $b = 0$ *either* $f(x) \equiv 0$ (*if* $a \neq 1$) *or* $f(x)$ *is arbitrary* (*if* $a = 1$) and that *for* $a = 0$ *we have* $f(x) \equiv 0$ whatever b may be. Thus in what follows, *we suppose* $b \neq 0$, $a \neq 0$.

How we proceed depends on whether the variables are *complex* or *real*. In the former case we multiply (1) by $a^{-1-x/b}$ and obtain

$$a^{-x/b}f(x) = a^{-(x+b)/b}f(x + b),$$

so that

$$p(x) = a^{-x/b}f(x)$$

is a periodic function of period b. As

$$f(x) = a^{x/b}p(x) \tag{2}$$

satisfies equation (1) whatever periodic function (with period b) $p(x)$ may be, we have shown, that (2) *is the most general complex solution of* (1) *if* $p(x)$ *is an arbitrary periodic function with period* b.

On the other hand, as $a^{x/b}$ is a continuous (multivalued) function, (2) *is the general continuous complex solution of* (1) *if* $p(x)$ *is an arbitrary continuous function with period* b.

If now $a, b, x, f(x)$ are *real* then we multiply (1) by

$$a^{-1}|a|^{-x/b}(sga)^{-[x/b]} \; ([t] = n \text{ if } n \leqq t < n + 1, n \text{ an integer};$$

$$sga = 1 \text{ if } a > 0,\ sga = -1 \text{ if } a < 0):$$

$$|a|^{-x/b}(sga)^{-[x/b]}f(x) = a^{-1}|a|^{-x/b}(sga)^{-[x/b]}f(x + b)$$

$$= |a|^{-(x+b)/b}(sga)^{-[(x+b)/b]}f(x + b)$$

thus

$$p(x) = |a|^{-x/b}(sga)^{-[x/b]}f(x)$$

is periodic with period b. As

$$f(x) = |a|^{x/b}\,(sga)^{[x/b]}p(x) \tag{3}$$

satisfies (1) whatever periodic function (with period b) $p(x)$ may be, we have shown that (3) *is the most general real solution of* (1), $p(x)$ *being an arbitrary periodic function of period* b. As to continuity, one sees that at points different from 0, $\pm b$, $\pm 2b$, ..., and also at these points from the right, $f(x)$ is continuous if and only if $p(x)$ is

continuous at $(0, b)$. As to continuity from the left at the points $x = nb$ (n an arbitrary integer)

$$\lim_{x \to nb-0} f(x) = |a|^n (sga)^{n-1} \lim_{x \to b-0} p(x)$$

and

$$f(nb) = |a|^n (sga)^n \, p(0);$$

thus $f(x)$ is continuous at these points if and only if

$$\lim_{x \to b-0} p(x) = sga\, p(0). \tag{4}$$

Thus we have shown that (3) *is the general real continuous solution of* (1) *if* $p(x)$ *is an arbitrary periodic function of period* b, *continuous in* $(0, b)$ *and satisfying* (4).

This can also be written in another form: If we define

$$r(x) = (sga)^{[x/b]} p(x)$$

then (3) *can be replaced by*

$$f(x) = |a|^{x/b} r(x),$$

where $r(x)$ *satisfies*

$$r(x + b) = sga \, . \, r(x)$$

and $f(x)$ *is continuous if and only if* $r(x)$ *is continuous.*

One can solve the problem also by *construction:* As (1) shows immediately, for $b \neq 0$ *the general solution can be got by drawing* $f(x)$ *in* $(0, b)$ *arbitrarily and continuing the function so that it satisfies*

$$f(x) = a^{[x/b]} f(x - [x/b]b),$$

and the function thus constructed is *continuous if and only if it was taken continuous in* $[0, b)$ *and* $\lim_{x \to b-0} f(x) = af(0)$.

Finally I remark that I posed 5 years ago the problem of determining all real solutions of the more general equations

$$f(ax + b) = cf(x) + d$$

which was solved also by A. Császár, M. Hossty and J. Surányi (Mat. Lapok 5 (1954), 48; 6 (1955) 353–355; 7 (1956), 127, 144.)

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Editorial note. A solution was also received from R. O. Davies.

2909. Approximations to square roots

If r is an approximation to the square root $r + h$ of a, then we can obtain a closer approximation, by using the identity

$$(r + h - r)^2 = (r + h)^2 - 2r(r + h) + r^2$$

and using the fact that h is small compared with r. Neglecting h^2 and replacing $(r+h)^2$ by its value a, we find

$$0 = r^2 - 2r(r+h) + a$$

whence

$$r + h = \frac{r^2 + a}{2r}, \qquad (1)$$

which is, of course, the expression obtained by applying the usual Newton approximation.

We can, however, go further with this method of derivation, for

$$(r + h - r)^3 = (r+h)^3 - 3(r+h)^2 r + 3(r+h)r^2 - r^3$$

Putting $(r+h)^2 = a$, we have

$$0 = r^3 - 3r^2(r+h) + 3ra - (r+h)a$$

whence

$$r + h = r\left(\frac{3a + r^2}{a + 3r^2}\right) \qquad (2)$$

which is a better approximation than (1).

It is possible to go a stage further still along the same lines, treating in the same manner the expansion of $(r + h - r)^4$ from which is obtained the formula

$$r + h = \frac{a^2 + r^2(6a + r^2)}{4r(a + r^2)} \qquad (3)$$

but this is identical with two applications of (1).

An example will show the accuracy of approximations from these three formulae, when used to determine $\sqrt{10} = 3{\cdot}162278$, with $r = 3$, say. Formula (1) gives $r + h = \dfrac{19}{6} = 3{\cdot}16667$ and the next approximation is $\dfrac{721}{228} = 3{\cdot}162281$. Formula (2) gives $r + h = \dfrac{117}{37} = 3{\cdot}162162$ and formula (3) gives 721/228 again, as previously observed. For practical purposes, recurrence formula (2) gives a close approximation with little computation.

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2910. Contracting Bernoulli's iteration and recurrence relations

To solve a polynomial equation, for example,

$$x^3 - x - 1 = 0 \qquad (1)$$

by Bernoulli's method, one has simply to calculate $f(3)$, $f(4)$, $f(5)$, ... using the recurrence formula

$$(E^3 - E - 1)f(x) = 0 \tag{2}$$

giving $f(0)$, $f(1)$, $f(2)$ any set of values, (but preferably $1, \xi, \xi^2$ where ξ is a rough value of the dominant root of equation (1)) and then take $f(n+1)/f(n)$ as an approximation for the dominant root of the equation. As Hildebrand [1, p. 461] shows, the convergence of Bernoulli's iteration for equation (1) is slow; and so to get a good estimate of the root one may have to calculate (say) $f(49)/f(48)$, which would necessitate forty-seven iterations. The following scheme contracts the process into far fewer steps.

From equation (2) we deduce the following equivalences:

$$\begin{aligned}
E^3 &= E + 1 \\
E^4 &= E^3 . E = E^2 + E \\
E^6 &= (E^3)^2 = E^2 + 2E + 1 \\
E^{12} &= (E^6)^2 = E^4 + 4E^2 + 1 + 4E + 2E^2 + 4E^3 \\
&= 7E^2 + 9E + 5 \\
E^{24} &= 200E^2 + 265E + 151 \\
E^{48} &= 170625E^2 + 226030E + 128801 \\
E^{49} &= E^{48} . E \\
&= 226030E^2 + 299426E + 170625
\end{aligned}$$

Therefore,

$$\frac{f(49)}{f(48)} = \frac{226030f(2) + 299426f(1) + 170625f(0)}{170625f(2) + 226030f(1) + 128801f(0)} = 1{\cdot}324717973 \ldots$$

putting $f(0), f(1), f(2) = 1, 1{\cdot}3, 1{\cdot}69$. This value differs from Hildebrand's ten decimal place value $1{\cdot}3247180428$ [1, p. 444] only in the eighth decimal place. Of course, an independent estimate of the error can be made by comparing $f(49)/f(48)$ with $f(50)/f(49)$.

This scheme of contraction can be applied to the solution of not only linear difference equations with constant coefficients but also to those with periodic coefficients, as the latter type can be reduced to the former [2].

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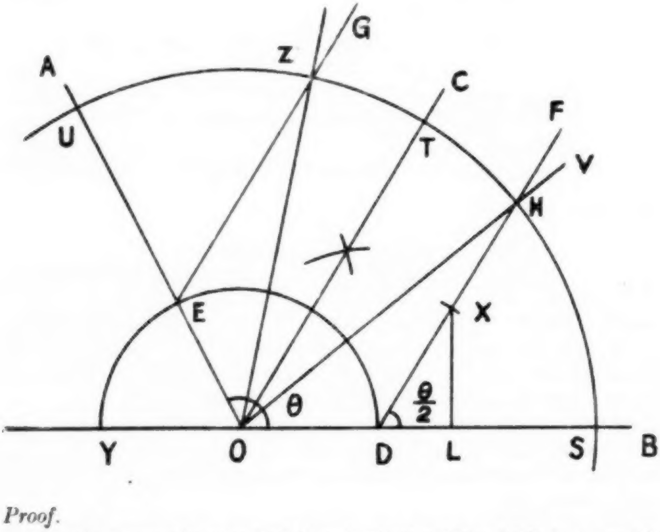
2911. Approximate trisection of an angle

There are a number of methods of approximate trisection of an angle in use. I have made an attempt which gives very close results for angles between 0 and 90 degrees.

Construction.

1. Draw the angle AOB
2. Bisect it by the line OC (Drawing complete arc of 180 D. DEY)
3. Draw DF and EG parallel to OC
4. Mark off $DX = OD$
5. Draw perpendicular XL on OB
6. Take YL in compass and taking O as centre draw STU, then angle VOB is the required approximately trisected angle with the two other parts as shown in diagram.



Proof.

If angle is exactly trisected chord SH should be $2OD(2 + \cos \frac{1}{2}\theta)$ $\sin \theta/6$. As construction shows

$$YL = 2OD + OD \cos \tfrac{1}{2}\theta = OD(2 + \cos \tfrac{1}{2}\theta).$$

Further the condition of trisection gives chord SH = chord ZH = chord $ED = 2OD \sin \frac{1}{2}\theta$. Therefore if the angle is exactly trisected $2(2 + \cos \frac{1}{2}\theta) \sin \theta/6 = 2 \sin \frac{1}{2}\theta$ or $(2 + \cos \frac{1}{2}\theta) \sin \theta/6 = \sin \frac{1}{2}\theta$. In fact this equality does not hold but approximately. I

have calculated for both sides of the equation for angles ranging from 0 to 180 degrees. The positive value of $(2 + \cos \frac{1}{2}\theta) \sin \theta/6 - \sin \frac{1}{2}\theta$ is in fact very small and can be taken as a measure of closeness.

Angle θ in degrees	$(2 + \cos \frac{1}{2}\theta) \sin \theta/6$	$\sin \frac{1}{2}\theta$	Mod. Difference
30	·2586	·2588	·0002
60	·4975	·5000	·0025
90	·7006	·7071	·0065
120	·8550	·8660	·0110
150	·9546	·9659	·0113
180	1·0000	1·0000	·0000

The above table shows how closely the derived equation holds for angles lying between 0 and 90 degrees.

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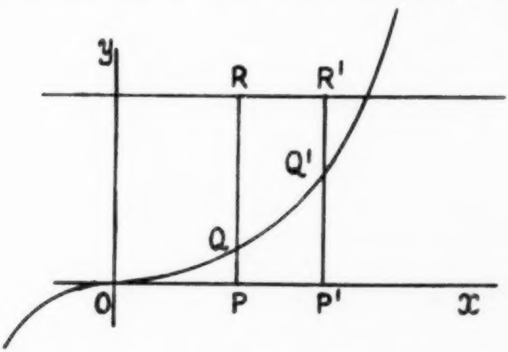
2912. On a proof of Desargues' theorem for a pencil of conics

1. A proof of Desargues' theorem, that the conics of a pencil meet



a fixed line in pairs of points of an involution, is often given in a form such as this. 'Let P be any point of the line. Then there is a unique conic of the pencil through P, and this meets the line again at P', say. Then P and P' are related by a symmetrical (1,1) correspondence, so (P, P') is a pair of an involution'.

The dangers of this 'proof' are not often appreciated. I should like to give two examples to show how this type of argument, which we call (A), can lead to false results. Similar examples have been

given before, but since (A) is still commonly used the point is worth making again.

2. (i) The diagram shows the curve $y = x^3$, the line $y = 1$, and two ordinates. Now if P and P' are chosen on OX so that $PQ = Q'R'$, taking account of sign, it is easily seen that P and P' are related by a symmetrical (1,1) correspondence, so by (A) (P,P') is a pair of an involution. However, this result is false.

The fallacy is that the relation between P and P' is (1,1) only if we confine ourselves to *real* points. It is necessary that the relation should be (1,1) for *complex* points.

(ii) The second example deals with complex points, and so a diagram of the line will be inadequate. However, we may make use of the Argand diagram: if z is the coordinate of a complex point P, referred to a suitable origin on the line, and $z = x + iy$, we represent P by the point (x, y).

Let P and P' be two (complex) points on a line such that the points which represent them on the Argand diagram are symmetrical with respect to the real axis. Then P and P' are related by a symmetrical (1,1) correspondence, so by (A) (P,P') is a pair of an involution. This result is also false.

Here the fallacy arises from the fact that the (1,1) correspondence is not *algebraic*, that is, z and z' are not related by an algebraic equation.

3. The examples in section 2 show that if (A) is to be made rigorous we must ensure that the correspondence is algebraic and (1,1) in complex numbers. Even then, we are still implicitly using the theorem that such a correspondence is a projectivity, and therefore an involution if it is symmetrical.

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2913. The nine-point conic of a convex quadrangle

If Q and R are points on a conic whose centre is O, there are points of the conic inside the triangle OQR only if the conic is a hyperbola and Q and R are on the same branch. This is the clue to a solution of an exercise (8.9, Ex. 7) in Coxeter's *Real Projective Plane*, and it is at Professor Coxeter's request that I have written out the following proof for the *Gazette*. The inhibiting effect of a warning that a problem is not easy is only appreciated in retrospect, when one has made up one's mind to disregard the warning.

Let $ACBD$ be a convex quadrangle, so lettered that if the diagonal points (AB, CD), (AC, BD), (AD, BC) are P, Q, R, then P is inside the quadrangle, the line CD passes between Q and R, and D is the nearer of the points C, D to the line QR. Let the harmonics of Q for AC and BD be Q_A and Q_B, and those of R for AD and BC be R_A and R_B; denote the midpoints of AC and BD by M_A and M_B, and those

of AD and BC by N_A and N_B, and denote the common midpoint of M_AM_B and N_AN_B by O.

Since D is between A and R, the midpoint N_A is between A and R_A, and similarly N_B is between C and R_B; hence O is inside the quadrangle AR_AR_BC. Similarly O is inside the quadrangle CQ_AQ_BB. Hence O is inside CQ_APR_B, and therefore P is inside the triangle OQR It follows that the conic with centre O through P, Q, R, which is the nine-point conic of $ACBD$, is a hyperbola, and that the diagonal points are all on the same branch.

E. H. NEVILLE

2914. Frequency and probability of a given "points-count" in bridge hands

Consider

$$f(x, y) \equiv (1 + yx^0)^{36}(1 + yx^1)^4(1 + yx^2)^4(1 + yx^3)^4(1 + yx^4)^4.$$

If each bracket represents a card, the "one" the process of not choosing

n	Frequency	Probability
0	2,310,789,600	0·0036396
1	5,006,710,800	0·0078842
2	8,611,542,576	0·0135612
3	15,636,342,960	0·0246236
4	24,419,055,136	0·0384544
5	32,933,031,040	0·0518619
6	41,619,399,184	0·0655410
7	50,979,441,968	0·0802809
8	56,466,608,128	0·0889219
9	59,413,313,872	0·0935623
10	59,723,754,816	0·0940511
11	56,799,933,520	0·0894468
12	50,971,682,080	0·0802687
13	43,906,944,752	0·0691433
14	36,153,374,224	0·0569332
15	28,090,962,724	0·0442368
16	21,024,781,756	0·0331092
17	14,997,082,848	0·0236169
18	10,192,504,020	0·0160508
19	6,579,838,440	0·0103617
20	4,086,538,404	0·0064354
21	2,399,507,844	0·0037787
22	1,333,800,036	0·0021004
23	710,603,628	0·0011190
24	354,993,864	0·0005590
25	167,819,892	0·0002643
26	74,095,248	0·0001167
27	31,157,940	0·0000491
28	11,790,760	0·0000186
29	4,236,588	0·0000067
30	1,396,068	0·0000022
31	388,196	0·0000006
32	109,156	0·0000002
33	22,360	$3·5 \times 10^{-8}$
34	4,484	$7·1 \times 10^{-9}$
35	624	$9·8 \times 10^{-10}$
36	60	$9·4 \times 10^{-11}$
37	4	$6·3 \times 10^{-12}$

that card, the power of x the "points count" of that card, and "y" the process of choosing the card, then it can be seen that $f(x, y)$ represents a pack of cards, and the coefficient of $y^{13}.x^n$ is the frequency of n points in a hand of 13 cards (ace, king, queen, jack counting 4, 3, 2, 1 points respectively).

The sum $635{,}013{,}559{,}600 = {}^{52}C_{13}$, the number of ways of choosing a hand from the pack (this is a check on the calculation).

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2915. The Simson line and the cardioid

In Note 2832, *Gazette*, Vol. XLIII, no. 344, May 1959, p. 123, R. K. Guy, after recalling the well known relation between the Simson line and the deltoid, mentions a connexion between that line and the cardioid.

In *Mathesis*, 1940, p. 295, I gave another such connexion:

When a given triangle rotates about its circumcentre, the Simson line of a fixed point on the circumcircle envelopes a cardioid.

If ABC is the position of the triangle for which the Simson line of the fixed point P is parallel to the Euler-line of ABC and if G and O are the centroid and the circumcentre of ABC, then the cusp and the singular focus of the cardioid are the mid-points of PG and PO.

The given proof used complex coordinates.

In the same paper, I mentioned another connexion between the Simson line and the cardioid:

When a given triangle rotates about its circumcentre, the line drawn through the orthocentre and forming a constant angle with the Simson line of a fixed point on the circumcircle envelopes a cardioid.

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2916. Integral solutions of the equations $ax_r - (a+1)x_{r+1} = 1$ $r = 1, 2, \ldots, n$.

Writing the equations in the form

$$ax_r - (a+1)x_{r+1} = (a+1) - a$$

we have

$$at_r = (a+1)t_{r+1}$$

where $t_r = x_r + 1$, and therefore $a^n t_1 = (a+1)^n t_{n+1}$ and since a, $a+1$ have no common factor, it follows that

$$t_1 = k(a+1)^n, \quad t_{n+1} = ka^n$$

and since

$$a^r t_1 = (a+1)^r t_{r+1}$$

therefore

$$t_{r+1} = ka^r(a+1)^{n-r}.$$

Hence the general solution is

$$x_{r+1} = -1 + ka^r (a+1)^{n-r}.$$

(The author is indebted to the Editor for the foregoing proof.)

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2917. The general solution in integers of the system of equations:

$$ax_r - bx_{r+1} = 1$$

$r = 1, 2, \ldots, n$, $(a, b) = 1$.

1. If $x_r = X_r$, $r = 1, 2, \ldots, n+1$ is the general solution in integers of the system of equations

$$ax_r - bx_{r+1} = 1, \quad r = 1, 2, \ldots, n,$$

then the general solution of the system

$$ax_r - bx_{r+1} = c, \quad r = 1, 2, \ldots, n,$$

where $(a, b, c) = 1$ is

$$x_{r+1} = cX_{r+1} + ka^r b^{n-r}, \; r = 0, 1, 2, \ldots, n.$$

For

$$ax_r - bx_{r+1} = acX_r - bcX_{r+1}$$

whence, writing $t_r = x_r - cX_r$, we have

$$at_r = bt_{r+1}$$

and so, as in the previous note,

$$t_{r+1} = ka^r b^{n-r}.$$

2. The general solution of the system of equations

$$ay_r - by_{r+1} = 1, \quad r = 1, 2, \ldots, n, n+1$$

is

$$y_{r+1} = \alpha + (\alpha - \beta)bX_{r+1} + ka^r b^{n-r+1}, \quad 0 \leqq r \leqq n,$$

$$y_{n+2} = \beta + a(\alpha - \beta)X_{n+1} + ka^{n+1}$$

where $x = \alpha$, $y = \beta$ is *any* solution of the equation

$$ax - by = 1.$$

For

$$ay_r - by_{r+1} = a\alpha - b\beta$$

so that

$$a(y_r - \alpha) = b(y_{r+1} - \beta)$$

and therefore

$$y_r - \alpha = k_r b, \quad y_{r+1} - \beta = k_r a$$

and so

$$ak_r - bk_{r+1} = \alpha - \beta, \quad r = 1, 2, \ldots, n,$$

of which the general solution, as we have seen, is

$$k_{r+1} = (\alpha - \beta)X_{r+1} + ka^r b^{n-r}, \quad r = 0, 1, 2, \ldots, n,$$

whence

$$y_{r+1} = \alpha + (\alpha - \beta)bX_{r+1} + ka^r b^{n-r+1}, \quad r = 0, 1, 2, \ldots, n,$$

and

$$y_{n+2} = \beta + (\alpha - \beta)aX_{n+1} + ka^{n+1}.$$

3. We consider now the system of equations

$$ax_r^n - bx_{r+1}^n = 1, \quad r = 1, 2, \ldots, n.$$

It follows from §2 that the general solution is given by the recurrence relations

$$x_{r+1}^{n+1} = \alpha + (\alpha - \beta)bx_{r+1}^n + ka^r b^{n-r+1}, \quad n \geqq r \tag{3.1}$$

and

$$x_{n+2}^{n+1} = \beta + (\alpha - \beta)ax_n^{n+1} + ka^{n+1}. \tag{3.2}$$

From 3.1 it follows that

$$x_{n+1}^{r+n+1} = \sum_{s=0}^{n} \alpha \{(\alpha - \beta)b\}^s + \{(\alpha - \beta)b\}^{n+1} x_{r+1}^r$$
$$+ ka^r b^{n+1} \sum_{s=0}^{n} (\alpha - \beta)^s$$

and from 3.2 that

$$x_{n+1}^n = \beta \sum_0^{n-1} \{(\alpha - \beta)a\}^s + ka^n \sum_0^{n-1} (\alpha - \beta)^s$$
$$= \beta \frac{a^n \delta^n - 1}{a\delta - 1} + ka^n \frac{\delta^n - 1}{\delta - 1}, \quad \delta = \alpha - \beta.$$

Hence the general solution of the system of equations

$$ax_r - bx_{r+1} = 1, \quad r = 1, 2, \ldots, n,$$

is

$$x_{r+1} = \alpha\left[\frac{(b\delta)^{n-r} - 1}{b\delta - 1}\right] + (b\delta)^{n-r}u_r + ka^r b^{n-s}\left[\frac{\delta^{n-r} - 1}{\delta - 1}\right],$$
$$r \leqq n - 1,$$

$$x_{n+1} = u_n$$

where

$$u_r = \beta\left[\frac{(a\delta)^r - 1}{a\delta - 1}\right] + ka^r\left[\frac{\delta^r - 1}{\delta - 1}\right], \quad \delta = \alpha - \beta.$$

and $x = \alpha$, $y = \beta$ is any solution of the equation $ax - by = 1$.

R. L. Goodstein

2918. On square sums of squares

The problem of finding three integers a, b, c such that $a^2 + b^2$, $a^2 + c$, $b^2 + c$, and $a^2 + b^2 + c$ are all squares was proposed as *Mahatma's* problem No. 78, in the Journal of the A.M.A. July 1949.

The smallest known solution $a = 124$, $b = 957$, $c = 13852800$ was given by J. Peacock, J. Hancock and N. A. Phillips in the September number of the same Journal. In this note I give a formula for an infinity of solutions to the problem, which yields the above smallest set of values for a, b, and c.

Since $a^2 + b^2$ is a square, we have $a = 2uv$, $b = u^2 - v^2$; writing

$$a^2 + c = p^2, \quad b^2 + c = q^2$$

we see that

$$p^2 + b^2 = q^2 + a^2$$

and each is a square, so that

$$p = 2u_1v_1,$$

with

$$u_1^2 - v_1^2 = u^2 - v^2$$

and

$$q = u_2^2 - v_2^2$$

with

$$u_2v_2 = uv$$

and

$$u_1^2 + v_1^2 = u_2^2 + v_2^2.$$

Thus the problem reduces to finding a general solution of the equations

$$\begin{aligned} u_1^2 + v_1^2 &= u_2^2 + v_2^2 \\ u_1^2 - v_1^2 &= u^2 - v^2 \\ u_2v_2 &= uv. \end{aligned}$$

These equations are satisfied by

$$\begin{aligned} u &= \tfrac{1}{2}(x - 1)(y^2 + xy - x^2), & v &= \tfrac{1}{2}(x + 1)(y^2 - xy - x^2) \\ u_1 &= \tfrac{1}{2}(x - 1)(y^2 + xy + x^2), & v_1 &= \tfrac{1}{2}(x + 1)(y^2 - xy + x^2) \\ u_2 &= \tfrac{1}{2}(x + 1)(y^2 + xy - x^2), & v_2 &= \tfrac{1}{2}(x - 1)(y^2 - xy - x^2) \end{aligned}$$

where $y = \frac{1}{2}(x^2 + 1)$.

Hence writing $x = 2t + 1$, $y = 2t^2 + 2t + 1$, and

$$\begin{aligned} U &= t(t + 2)(2t + 1)^2 + (t + 1)^2, \\ V &= (t^2 - 1)(2t + 1)^2 + t^2 \\ U_1 &= (t^2 + 2t + 2)(2t + 1)^2 + (t + 1)^2, \\ V_1 &= (t^2 + 1)(2t + 1)^2 + t^2 \end{aligned}$$

we have

$$a = 2t(t+1)UV, \ b = t^2U^2 - (t+1)^2V^2,$$
$$p = 2t(t+1)U_1V_1, \ c = p^2 - a^2,$$

and $t = 1$ gives $a = 124$, $b = 957$, $c = 2^4 . 7^4 . 19^2 - 124^2$.

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2919. Observations on Prime Numbers

Let p_n denote the nth prime number and

$$q_k = \sum_{i=1}^{k} p_i + \tfrac{1}{2}[5 - (-1)^k].$$

For $k = 1, 2, \ldots, 20$, q_k is a prime number; the corresponding values of q_k are: 5, 7, 13, 19, 31, 43, 61, 79, 103, 131, 163, 199, 241, 283, 331, 383, 443, 503, 571, 641.

The numbers n, $n + 30$, $n + 60$, $n + 90$ for $n = 4637579$ and 6248969 are consecutive primes (it is known that if there exists an arithmetical progression of five consecutive primes, then its difference is divisible by 30).

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2920. Finite difference notation

The Mathematical Association's Report "The Teaching of Algebra in Sixth Forms", in dealing with Finite Series (p. 72) states that the equation

$$\Delta r^{(k)} = k \,.\, r^{(k-1)} \tag{i}$$

is not valid for negative k, unless negative factorial powers are defined as

$$r^{(-k)} = \frac{1}{(r+k-1)^{(k)}}. \tag{ii}$$

This implies that (ii) is the only relationship for which (i) is valid for negative k, and in this respect the Report's statement is incorrect.

Let

$$r^{(-k)} = \frac{1}{(r+k+c)^{(k)}} \tag{iii}$$

Then

$$\Delta r^{(-k)} = \frac{1}{(r+k+c+1)^{(k)}} - \frac{1}{(r+k+c)^{(k)}}$$
$$= \frac{-k}{(r+k+c+1)^{(k+1)}}$$
$$= (-k) \,.\, r^{(-k-1)} \tag{iv}$$

So that definition (iii) makes (i) valid for negative k, when c is a positive integer, zero or a negative integer.

It remains to consider which is the most appropriate, logical and useful value to take for c. The Report follows Boole [1] and Steffensen [2] (who presumably also took his notation from Boole), putting $c = -1$. It is, in this connection, interesting to note that, according to Jordan [3], special notation for negative factorial powers was first introduced by Vandermonde [4], who used:—

$$[x]^n = x(x-1)(x-2)\dots(x-n+1),$$

and

$$[x]^{-n} = 1/(x+1)(x+2)\dots(x+n)$$

i.e., Vandermonde used (iii) with c equal to *zero*. Aitken [5], Milne-Thomson [6], Freeman [7], Hogben [8], Richardson [9] and Buckingham [10] have followed Vandermonde's notation, and put:—

$$r^{(-k)} = \frac{1}{(r+k)^{(k)}}. \tag{v}$$

This notation is more logical than that used by Boole (which is presumably the reason why so many modern authors have adopted—or, rather, re-adopted—it). Since $r^{(k)}$ is defined as $r(r-1)(r-2)\dots(r-k+1)$, it follows that

$$r^{(k)}(r-k) = r^{(k+1)} \tag{vi}$$

Using Boole's definition (ii), (vi) is only valid for k a positive integer or zero. Using Vandermonde's original definition (v), (vi) is valid for k a positive integer, zero or a negative integer. More generally, Vandermonde's definition is valid, and Boole's definition is not valid, for the following relationship, provided that division by zero is not involved, where k and n have positive integral, zero or negative integral values:—

$$r^{(k)}(r-k)^{(n)} = r^{(k+n)}. \tag{vii}$$

It can be shown that only by putting $c = 0$ in (iii) can we arrive at a definition for $r^{(-k)}$ which will satisfy relationships (vi) and (vii) for negative k. Using (iii) in conjunction with (vii) we have:—

$$r^{(-k)}(r+k)^{(n)} = r^{(-k+n)}$$

i.e., using (iii):—

$$\frac{(r+k)^{(n)}}{(r+k+c)^{(k)}} = \frac{1}{(r+k-n+c)^{(k-n)}}$$

Multiply both sides by $(r+k+c)^{(k)}$. Then:—

$$(r+k)^{(n)} = (r+k+c)^{(n)}$$

so that c must equal *zero*.

Vandermonde's definition can be shown to be more logical than Boole's in another connection. Since

$$r^{(k)} = r(r-1)(r-2)\dots(r-k+1)$$

we have

$$r^{(k)} = \frac{r(r-1)(r-2)\dots(r-k+1)(r-k)\dots 3.2.1}{(r-k)\dots 3.2.1}$$

$$= \frac{r!}{(r-k)!} \qquad \text{(viii)}$$

Put $-k$ for k, and we have

$$r^{(-k)} = \frac{r!}{(r+k)!} \qquad \text{(ix)}$$

$$= \frac{r(r-1)\dots 3.2.1}{(r+k)(r+k-1)\dots(r+1)r(r-1)\dots 3.2.1}$$

$$= \frac{1}{(r+k)(r+k-1)\dots(r+1)}$$

$$= \frac{1}{(r+k)^{(k)}}$$

which is definition (v), so that this definition makes (viii) valid for k a positive integer, zero or a negative integer, whereas definition (ii) makes it invalid for a negative integer.

In view of the above, it would be interesting to know why Boole adopted the notation he did. It does not appear to have any compensating advantages.

Boole's *Treatise* was published a hundred years ago, and possibly a more modern, and more elementary, treatment, such as that given in Richardson's "An Introduction to the Calculus of Finite Differences", would be a more suitable choice of reference for Chapter 8 of the Report.

In the section of the Report dealing with Summation (p. 74), the following two standard summation formulae are given:—

$$\sum_{r=1}^{n} r^{(k)} = \frac{1}{k+1}(n+1)^{(k+1)} \quad \text{if} \quad k > 0. \qquad \text{(x)}$$

$$\sum_{r=1}^{n} r^{(-k)} = \frac{1}{1-k}\left\{(n+1)^{(1-k)} - 1^{(1-k)}\right\} \quad \text{if} \quad k > 1. \qquad \text{(xi)}$$

It is perhaps of interest to note that we may express (x) and (xi) in one general formula, which also includes the case $k = 0$, not included above:—

$$\sum_{r=1}^{n} r^{(k)} = \frac{1}{k+1}\left\{(n+1)^{(k+1)} - 1/(-k)!\right\} \qquad \text{(xii)}$$

Here k may take any integral value unless division by zero is involved, i.e., it can be a positive integer, zero or any negative integer except -1. Both Vandermonde's and Boole's definitions will make (xi) valid, but only Vandermonde's definition will make (xii) valid, in the form given when k is negative.

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2921. A hypothesis on prime numbers

Mersenne numbers (M_n) are of the form $2^n - 1$ and are the only numbers of the form $x^n - 1$ which can be prime, since $x - 1$ is a factor of this expression. $2^n - 1$ is known to be prime by using Lucas's Test when $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281$ and 3217 (the latter being the largest known prime number, containing 969 digits). $2^{(2^n-1)} - 1$ is prime for $n = 2, 3, 5$ or 7, $2^p - 1$ when $p = 2^{(2^n-1)} - 1$ is prime when $n = 2$ or 3 and $2^{(2^p-1)} - 1$ when $p = 2^{(2^n-1)} - 1$ is prime when $n = 2$. So far, no case has been found such that these four series are different.

I therefore concluded that if $2^n - 1$ is prime, then $2^{(2^n-1)} - 1$ is also prime. But $2^n - 1$ is only one example of the general expression $(x^n - a^n)/(x - a)$ and I therefore arrived at the following hypothesis which I have not yet found to be incorrect, but seven cases where it is correct.

Hypothesis

If a is any positive integer and x, p_n and $\dfrac{x^{p_n} - a^{p_n}}{x - a} = p_{n+1}$ are prime, then $p_{n+2}, p_{n+3}, \ldots$ are also prime.

If x is composite, then p_3 is composite for $a = 3$, $x = 8$, $n = 2$ and $p_2 = 11$. $p_3 = \dfrac{8^{11} - 3^{11}}{8 - 3} = 1{,}717{,}951{,}489 = (23)(67)(1{,}114{,}829)$. If this hypothesis were correct, an infinite number of large primes could immediately be found. The following table gives values of p_n and those which are known to be prime are underlined.

x	a	p_1	p_2	p_n
2	1	2	3	$p_3 = \underline{7}$.
				$p_4 = \underline{127}$.
				$p_5 = \underline{170{,}141{,}183{,}460{,}469{,}231{,}731{,}687{,}303{,}715{,}884{,}105{,}727}$.
3	2	2	5	$p_3 = \underline{211}$.
5	2	2	7	$p_3 = \underline{25{,}999}$
3	1	3	13	$p_3 = \underline{797{,}161}$
7	4	2	11	$p_3 = 657{,}710{,}813$.
3	2	3	19	$p_3 = 1{,}161{,}737{,}179$.
2	1	5	31	$p_3 = \underline{2{,}147{,}483{,}647}$.
7	6	2	13	$p_3 = 83{,}828{,}316{,}391$.
11	2	2	13	$p_3 = 3{,}835{,}856{,}903{,}971$.
11	6	2	17	$p_3 = 101{,}086{,}020{,}367{,}969{,}807$.
11	8	2	19	$p_3 = 20{,}338{,}325{,}083{,}779{,}563{,}473$.
5	1	3	31	$p_3 = 1{,}164{,}153{,}218{,}269{,}348{,}144{,}531$.

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2922. Some further remarks on moments of inertia

Those readers who found the paper by D. E. Rutherford in the February 1960 issue interesting may also be interested in the following further results found by similar methods. These results are about the existence of equimomental systems. Such systems are defined in the standard textbooks as having the same moment of inertia about any axis. It follows at once from Rutherford's second basic result (parallel axes) that such systems must have the same mass and the same centre of gravity and from the definition they must then have the same principal axes and principal moments there. Conversely, it is very easy to show that these properties are also sufficient to make the systems equimomental. Rutherford's fifth basic result about similar bodies then shows that if two systems are equimomental and we expand each system about a certain axis in a constant ratio, the new systems will again be equimomental.

The equimomental systems considered in dynamics usually consist of a system of a finite number of particles which is equimomental to a rigid body. The interesting question at once arises, how many particles are sufficient? For an r-dimensional body we must clearly have at least $r + 1$ particles, for Rutherford's first basic result shows, with a little argument, that a system of r particles can only be an equimomental system for a body of less than r

dimensions. The interesting cases are for $r = 1$, 2 or 3 but the argument can easily be generalized to any value of r. The principal result of this paper is that $r + 1$ particles are always sufficient. We shall prove this for $r = 3$. Notice firstly that all the moments and products of inertia of a body are specified if we specify the principal moments at the centre of gravity. We can then certainly find a uniform ellipsoid which has the same principal moments at that point, which we choose as the centre of the ellipsoid. (This is because we have to fit three principal moments and the ellipsoid has three disposable axes.) Now, by applying Rutherford's fifth result along the three axes of the ellipsoid, we can transform it into a sphere and to show that a system of four particles exists equimomental with the original body it is now only necessary to show that such a system exists equimomental with the sphere. It is very easy to verify that such a system does exist. One such consists of four equal masses regularly distributed on a concentric sphere of radius $a\sqrt{3/5}$. This then proves the result. Incidentally we have proved a little more, since we have shown that this best possible system is attainable with the masses of the $r + 1$ particles all equal. It is of interest to note that there will be an equimomental system for a rod of only two particles and it is very easy to see what this is.

The reader may find it interesting to use Rutherford's methods to prove that for the uniform ellipsoid there is an equimomental system of four equal particles which lie on a similar, and similarly situated, ellipsoid and which form the vertices of the maximum inscribed tetrahedron.

Most of the above results are to be found in Routh's *Rigid Dynamics* (London 1913, page 29) but the present argument is rather more direct.

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2923. Two problems

1. Find the general solution in positive integers of the equations:

$$\begin{cases} a + 3b = c + d = n^3 \\ a^3 + 3b^3 = c^3 + d^3. \end{cases}$$

2. Find integral solutions of the system of equations

$$\begin{cases} A_1^2 + A_2^2 + A_3^2 + A_4^2 = B_1^2 + B_2^2 + B_3^2 + B_4^2 = C_1^2 + C_2^2 + C_3^2 = D_1^2 + D_2^2 + D_3^2 \\ A_1^4 + A_2^4 + A_3^4 + A_4^4 = B_1^4 + B_2^4 + B_3^4 + B_4^4 \\ A_1 \,.\, A_2 \,.\, A_3 \,.\, A_4 = B_1 \,.\, B_2 \,.\, B_3 \,.\, B_4 \\ A_1 + A_2 + A_3 = A_4; \quad B_1 + B_2 + B_3 = B_4 \\ A_1 + A_2 + A_3 + A_4 + D_1 + D_2 + D_3 = B_1 + B_2 + B_3 + B_4 + C_1 + C_2 + C_3 \\ A_1^2 + A_2^2 + A_3^2 + A_4^2 + D_1^2 + D_2^2 + D_3^2 = B_1^2 + B_2^2 + B_3^2 + B_4^2 + C_1^2 + C_2^2 + C_3^2 \\ A_1^4 + A_2^4 + A_3^4 + A_4^4 + D_1^4 + D_2^4 + D_3^4 = B_1^4 + B_2^4 + B_3^4 + B_4^4 + C_1^4 + C_2^4 + C_3^4 \\ A_1^6 + A_2^6 + A_3^6 + A_4^6 + D_1^6 + D_2^6 + D_3^6 = B_1^6 + B_2^6 + B_3^6 + B_4^6 + C_1^6 + C_2^6 + C_3^6 \\ A_1^8 + A_2^8 + A_3^8 + A_4^8 + D_1^8 + D_2^8 + D_3^8 = B_1^8 + B_2^8 + B_3^8 + B_4^8 + C_1^8 + C_2^8 + C_3^8 \end{cases}$$

Gunzenhausen, Bavaria ALFRED MOESSNER

2924. The final problem

As a thank offering to the gods for release from the bondage of everlasting examinations, I bequeath the following to my successors.

Two conics are related so that there is one quadrilateral, (and hence an infinity) inscribed in *each* and circumscribed to the *other*.

Prove that there are two essentially distinct cases, the first typified in Euclidean geometry by two equal circles each through the centre of the other, and the second typified by the following figure:

A quadrilateral $ABCD$ is inscribed in a circle, centre O, so that AB is a diameter and CD is parallel to it, and such that a circle can be inscribed in the quadrilateral.

Prove also in the second case that if α is half the angle AOD, then

(i) $\sin \alpha = \frac{1}{2}(\sqrt{5} - 1) = 2 \sin \dfrac{\pi}{10}$

(ii) α is approximately $\tan^{-1} \pi/4$

(iii) the triangle AOD is similar to the mid section of the pyramid of Khufu (or Cheops), namely, approximately slant height: base: vertical height = 89: 110: 70.

(iv) Deduce an approximate construction for π.

$\left(\textit{Note}: \sqrt{\tfrac{1}{2}(\sqrt{5} - 1)} = \dfrac{1}{4}\pi + 0{\cdot}0007.\right)$

Auckland University College, New Zealand H. G. FORDER

GLEANINGS FAR AND NEAR

1947. (referring to George Bidder) "...When called upon to multiply 89 by 73 he would say instantly 6497 but on reflection he went through quite a lengthy process. He had in fact added the sum of the quotients of 80 by 70, 80 by 3, 9 by 70 and 9 by 3."

Times Educational Supplement for 4 Sept., 1959, p. 220. [Per Mr. A. T. F. Nice.]

REVIEWS

The Teaching of Mathematics in the Secondary Modern School. By Olive I. Morgan. Pp. 144. 10s. 6d. 1959. (George G. Harrap)

The author limits the scope of her book by the sub-title "A Practical Guide for the Non-mathematician" and also by an introduction which states that her choice of material is influenced by questions asked by teachers, particularly questions about the first and last years of modern school work. No doubt these questions are in part responsible for the emphasis on arithmetic.

The book is divided into four parts. Part One which is quite short, is rather curiously named "The Content of the Curriculum". It contains some interesting matter about the nature and aims of elementary mathematics teaching and about school organisation, but as a guide to the possible content of the curriculum and to the self-imposed limitations of the book itself it is disappointing.

Part Two, "The Foundations of Understanding", again strangely named, contains all the book has to say about measuring and geometry, mainly about their beginnings. The understanding of number is not discussed in this section.

Part Three, "The Tools of Arithmetic", occupies half the book. It covers familiar ground. The analysis of ways of computing which children may have acquired before entering a modern school is perhaps usefully informative for teachers dealing with revision, but the desirability of the various ways of working described is not appraised critically enough to form a teaching guide.

Part Four deals quite shortly with two problems of the modern school, the last year of school life and the backward pupil. It is helpful so far as it goes, but in dealing with the curriculum it is almost exclusively about familiar applications of arithmetic.

L. D. Adams

Counting and All That. By A. Monteith. Pp. 206. 15s. 1958. (George G. Harrap and Co. Ltd.)

This book about arithmetic for young children is by an experienced author who in a valuable first chapter surveys a variety of modes of dealing with her subject in schools. She states her belief in a "comprehensive" use of apparatus rather than a "selective" adherence to some particular type of didactic material. The part played by activity and experience is also discussed but the trend of the book as a whole is towards experience fitted to logical stages rather than towards teaching dovetailed into experience.

A sentence in the middle of the book seems to sum up the author's main intention: "Alongside much practical work there should be a course in arithmetic, pure and simple, taught with apparatus that shows the significance of each process." A careful analysis of this course of pure arithmetic illustrated by over a hundred beautifully

produced drawings of apparatus occupies the bulk of the book. Wise comments on other aspects of mathematics teaching are quite overshadowed by all this. The criticisms which follow are generally directed towards the contents of these main sections.

The apparatus illustrated is of very unequal value. For example, Figure 28 shows a mere device for setting sums; the use of bead bars for this trivial purpose might well cause confusion with their use for number composition as shown in Figure 20. Again some of the early apparatus would hardly be needed at all by abler children interested in number and its uses. On the other hand the sophisticated notions of decimal notation are wisely dealt with in detail and the blemish, if any, on the suggested use of apparatus for learning to compute with numbers greater than ten is that teaching is emphasised rather than experiment with apparatus by the learner.

The footnote for teachers (page 41) on zero as the origin in a directed number system might have been more valuable if it had also been shown how usages of directed number may fall within the experience of young children, as may other usages of the symbol "0".

The importance of activities involving money, weights and measures in giving number experience has been masked by the separation of a systematic approach to these topics into two late chapters. The references elsewhere are not sufficient to show what a great influence such practical experience can have on the number sections of the "Foundation Course" and on the "Counting Course" as described in the first and last Chapters.

L. D. A.

The Teaching of Arithmetic in Primary Schools. By L. W. Downes and D. Paling. Pp. xx, 500. 15/- in U.K. 1958. (Oxford University Press)

This is a comprehensive handbook for teachers concerned with Arithmetic in Primary Schools, whether in planning syllabuses and supervising what is done, or in dealing with day to day work in the classroom; such teachers are not necessarily mathematicians.

The authors state that a child's success in arithmetic depends largely on three things: firstly, a basis in his own experience which helps him to understand; secondly, a real understanding of what he is teaching on the part of the teacher; thirdly, a careful planning and grading of the work. Though the necessity for a basis of experience is frequently referred to the book is, perhaps unavoidably, concerned mainly with the second and third points. After three general chapters there are twenty-four which treat in detail all the arithmetic usually taught in English Primary Schools, and also, more briefly, percentages and the metric system: the various rules occur in the traditional order, and with a wealth of illustration and discussion of methods of teaching. At the end are two further chapters on providing equipment and making apparatus, and on organisation, testing and marking, and these are supplemented by three appendixes including a list of books for the mathematics library. The text is interspersed with illustrations of

apparatus, and diagrams of suggested ways of setting out work; there are also nine lists of "facts" and eleven charts or tables. Following ten of the chapters are analyses setting out in tabular form the suggested stages and steps in teaching addition, subtraction, multiplication, division, money, length, weight, capacity, time and fractions. At intervals between the chapters are fifteen articles in different type intended to give the teacher "something to think about," and to prepare him to understand the way in which the arithmetic he teaches has come into being. Finally there is an index occupying seven pages.

This very complete treatment of the arithmetic of the Primary School was prepared for teachers who would read it in English but would teach in one of the vernaculars of the Tropics: they would therefore have to assimilate it very thoroughly and prepare their own adaptations of the material. The importance of suiting the teaching to the local conditions of the class being taught is mentioned at intervals throughout the book but it is assumed that the teachers themselves will be familiar with English practice. True, mathematics is the same all over the world, but the experience of children when they begin to learn is surely very different, and it is to be regretted that almost the only kind of experience referred to is that planned by the teacher or based on his questioning, (p. 47). Nowhere is there any suggestion that a child may ask questions or find his own problems; the initiative is always with the teacher. Moreover, it is implied more than once that all the problems met with will be expressed in words, and there is a good deal of emphasis on the phrasing of questions in English textbooks which might, or might not, be susceptible of translation. Some of the language which is used with very small children tends to be irritating when addressed to teachers, and cannot be necessary when in any case the teacher must talk to his pupils in another tongue. Even for English teachers the word "share" is overworked, and it is surely inaccurate to speak of sharing something *between eight* boys (p. 212). Another word which is frequently misused in arithmetic textbooks, and is so here, is "altogether" in the sense of "together" or "in all". It is unfortunate, too, that the authors suggest that signs of operation are extraneous symbols whose placing can be determined to suit the school. In many parts of the book they seem to make it their motto that the teacher should be fore-warned of likely mistakes, but there is no warning against writing $8 - 17$ for $17 - 8$ or $4 \div 20$ for $20 \div 4$ and children certainly do make such mistakes.

The book is easy to read: it is produced in clear and varied type; there are frequent cross references, and footnotes are repeated if required again; each chapter is summarised at its end; the diagrams and other illustrations are placed where they can be seen when the relative paragraphs are being read; points likely to be criticised or to be controversial are the subject of disarming footnotes so that they do not delay the reader. A few printer's errors have been noted: a *c* for *e* at the foot of p. 351; the omission of an *8* on p. 389; wrong spacing of the ringed figures on p. 404. But in spite of criticisms the book is likely to prove a valuable contribution to teacher-training.

H. M. Cook

Problems in Arithmetic. Parts I, II, III. By H. B. Beech. 64 pages per book, 2/6d. each. Teacher's Book I 4/6d., II 4/6d., III 5/-. 1959. [A. & C. Black]

The three books in this series consist of questions graded according to the number of mechanical processes they contain. Questions in Book 1 contain one or two mechanical processes: those in Book 3 may include up to four processes. Three corresponding Answer Books contain the sums carefully worked and displayed on the basis of mechanical content.

Whether this type of question constitutes a "problem" is very doubtful. The essence of all problems is the mathematical relationships they contain. The number of mechanical processes, of whatever grade of difficulty, required for their solution bears no relation in many cases to the quality of thinking involved.

Furthermore, the segregation of questions of similar type under headings, e.g. "WEIGHTS AND MEASURES—Long Multiplication" tends to reduce the amount of reasoning required for their solution. Once the prototype is solved, the rest fall into place as mechanical repetitions. This may have a certain practice value but surely not in the solution of genuine mathematical problems.

The main use of these books would seem to be as supplementary material to a well classified Arithmetic course. The work in Book 2, for example, is classified under Capacity, Number, Length, Weight, Time, and Vulgar Fractions.

Reading matter is not always well graded and this in itself constitutes a problem for the less able children.

H. Chesterman

Geometrical Drawing. By G. Pearson. Pp. 128. 12s. 6d. 1957. (Oxford University Press)

This introductory text provides a basic course in geometrical drawing intended to cover the requirements of O-level G.C.E. syllabuses. It does, in fact, cover also some rather more advanced work and is, in general, both well-conceived and well-executed.

Approximately one-third of the book deals with plane geometrical drawing. The text is simple and the diagrams are exceptionally clear. Sometimes the student is liable to be misled concerning background theory. For example in the wording of that section dealing with the construction of any regular polygon on a side of given length the second method given is described as "entirely a geometrical construction" in contradistinction to the immediately preceding trial and error method. In fact, though the *construction* is geometrical the method has no Euclidean justification being only an approximation in the general case; this is even true for the regular pentagon for which a truly accurate construction would have been available.

The first portion of the book concludes with a consideration of sundry loci including such curves as the involute of a circle and the spiral of Archimedes—the latter locus being somewhat imprecisely defined as "the path of a point moving around and approaching a fixed point by equal amounts".

The remainder of the text consists of ten chapters on solid geometrical drawing. The author's scheme is to introduce orthographic projection first. Great emphasis is laid on the traces of planes (Chapter XIII) after which the harder work on orthographic projection is developed without a break. The discussion of isometric and oblique projections is left till the end of the book and the question of the timing of their presentation to the student is left to the teacher's discretion.

The book is well printed, clearly illustrated and contains a quite adequate supply of exercises.

J. KERSHAW

Mathematics for Higher National Certificate. By A. GEARY, H. V. LOWRY and H. A. HAYDEN. Pp. 336. 15s. 0d. 1957. (Longmans)

Many teachers will have used Part I of the authors' 'Advanced Mathematics for Technical Students' and some may have found (as did the reviewer) the going rather hard for the weaker H.N.C. students. The present volume is described in the preface as "more or less a shorter and easier version" of that earlier book.

It is unnecessary to say that the text is a sound workmanlike job which fully maintains the standard of preceding volumes by these authors and the fruit of the combined experience of this formidable team should not lightly be criticised. The exposition both in sequence and in detail is always economical, a factor of much importance in this class of work where time is so strictly limited; there are few teachers who would not find the text helpful in this regard. Perhaps one might sigh for a few novelties but we must realise the H.N.C. examination syllabuses have an inherent inertia making change or reform a slow business. It is refreshing to see the D-operator method of solving linear differential equations with constant coefficients set forth with a pleasant simplicity in a short final eight page chapter.

The book is well printed and there are plenty of exercises with answers at the end of the book. In checking a number of these not all, by any means, were found to be correct, a feature which makes the marking of "set work" based on the chapter end exercises quite interesting for the lecturer.

As students for H.N.C. are apt to prepare for their courses by clearing their minds of all previous learning an introductory chapter has been inserted containing basic essential mathematical preparation.

J. KERSHAW

An Outline of Elementary Geometry. By R. H. COBB and J. L. LEWIS. Pp. 42. 3s. 6d. 1959. (Basil Blackwell: Oxford)

This outline covers the Geometry needed for the Ordinary Level of the Oxford and Cambridge Board in a compact way. The large page (9 ins. by 7 ins.) allows the matter to be displayed clearly and related results to be visible together. Some 88 propositions or results are mentioned: full proofs are given of the dozen or so theorems asked for by the Board: there are about 100 examples. Apart from the authors'

purpose, the book would be useful as a "self help" or auxiliary text to O.L. candidates of other examining boards.

J. W. HESSELGREAVES

Mechanics. A New Introduction. By L. W. F. ELEN and R. MYERS. Pp. 304, 8s. 6d. Second edition, 1958. (Cleaver Hume Press)

This book is designed to cover the syllabus needed for the Statics, Dynamics and Hydrostatics at the Ordinary Level of the G.C.E. or for the First Year (S. 1.) classes in Technical Colleges. It would also serve as an introductory book for the Advanced Level G.C.E. course.

The text is concise and well arranged. The worked examples are very suitably chosen, the diagrams are attractive and of good size, and there is a directness and precision of wording which a student may well adopt. Within the chapters, sets of examples follow each main topic; each chapter ends with a set of miscellaneous examples covering all the topics of the chapter.

The provision of exercises (960) and diagrams (224) is almost too generous; the answers to the examples are well set out; the detail in which the contents are given avoids the need for an index. In their preface to this edition the authors say that the book has been in steady and latterly increasing demand. This is not surprising, as it is well written and produced, and good value for the money.

J. W. HESSELGREAVES

The Fundamentals of Mathematics. Revised Edition. By M. RICHARDSON. Pp. 507. 45s. 6d. 1958. (Macmillan, New York)

Previous editions of this book appeared in 1937, 1940 and 1941 but the present edition is so extensive a revision as to constitute a new book.

Teachers of mathematics are being constantly exhorted to include in their syllabuses some of the topics which concern present-day mathematicians and to give to their treatment of mathematics something of the "new" flavour. There are, however, very few books which offer any help or guidance at a level useful to the secondary teacher. Dr. Richardson is to be congratulated on producing this book which goes far toward fulfilling the needs of both pupil and teacher.

A brief section on deductive logic, truth and validity is followed by an account of the evolution of number systems and their theory as far as it affects the secondary school, forming a useful introduction to the first chapters of a Modern Algebra textbook and leading naturally to a discussion of the algebra of sets and logic and some of the applications of the theory. A short account of some unsolved problems (duplication, trisection, Fermat, the four colour problem) forms a diversion which might have been more valuable had some attempt been made to show their influence on some of the later developments of mathematics. Analytical geometry including a section on linear programming follows and again gives a foretaste of later developments. A study of algebraic polynomial functions follows to give the principal notions of the calculus and is followed by a chapter headed "Trigonometric Functions". A

section on probability and statistics which in the space of 43 pages touches matrices, game theory and correlation ends rather tamely with the histograms of distributions. The final chapters form a stimulating introduction to some of the problems of transfinite numbers, non-euclidean geometry and simple notions of mathematical systems and group theory. Each section has a well graded selection of exercises for the reader to test his understanding and has illustrative worked examples.

The great feature of the book is its freshness: not just a rearrangement of old topics but a new attack, a new attitude, a new point of view. There is reference to and portraits of men of mathematics to show how a topic was born and the subsequent treatment opens new vistas of applications to present problems in all fields of human endeavour, without allowing these applications to obscure mathematical developments but rather suggesting further developments. All this is achieved in a mathematical setting which demands little more than the techniques of the alternative syllabus. The book has its disappointments. Too many topics are raised and then left for follow-up elsewhere but there is a most comprehensive bibliography to each chapter which makes this follow-up easier. The chapter of the trigonometrical functions gives the conventional and traditional mensurational approach and stops short at a brief mention of the sine graph. 'e' is treated only as an example of a limit (the binomial or compound interest approach) but no idea of the importance of the exponential function ensues. As an introduction to mathematics many would quarrel with the book for not integrating such topics as simple equations and coordinate geometry more closely but this is not the purpose of the book. It is an eminently readable reappraisal of elementary mathematics which should form an excellent bridge to the sixth form. Many teachers who are wondering what to do with would-be entrants to Training Colleges could not do better than to read through this book with them. Most of all, one would recommend this book as a mathematics book which could be read.

C. Hope

Fluid Dynamics. By D. E. Rutherford. Pp. 226. 10s. 6d. 1959. (Oliver and Boyd)

This is an attractive, straightforward and elementary introduction to the basic mathematical concepts of Fluid Dynamics. The author rapidly passes in review the fundamentals of the classical theory of the perfect fluid, and then introduces us to the effects of compressibility and of viscosity.

Some very attractive features of this book are the description of Mackie's new amplification of the Hodograph method to solve problems with free stream lines and an account of irrotational methods for the solution of the potential equation in two dimensions.

The book is well adapted to first year students at University and is a welcome addition to the series of University Mathematical texts published by Oliver and Boyd.

G. Temple

Fallacies in Mathematics. By E. A. Maxwell. Pp. 95. 1959. 13s. 6d. (Cambridge University Press)

We all know how to prove that any triangle is isosceles, and how to expose the flaw in the argument by drawing an accurate figure. But do we all know how to deal with the contumacious opponent who maintains that his diagram is just as likely to be right as ours? If we have not got Pasch's axiom at our finger-tips, we had better buy Dr. Maxwell's book at once. He has not only made an interesting collection of fallacies, new and old, and exhibited the logical errors, he has in most cases traced these to their roots. The schoolboy at about A level, and his teacher, should find the book as useful as it is entertaining. The proper answer to the pupil who says "No doubt your method gives the right answer, but I still do not see what is wrong with my method" is a reference to basic principles and logical deductions, and it is this constant reference back to fundamentals which lifts Dr. Maxwell's volume far above the usual level of the "Mathematics for fun" type of book.

From the text, we may deduce that the author is a humane and experienced examiner, for some of his items have surely been acquired in the course of patient and toilsome efforts to decipher and unravel complications in examination scripts. We might also deduce his mathematical nihilism, from the glee with which he proves that there are practically no numbers or lengths, that a circle has neither inside nor radius, that there are no variables, no quadrilaterals, almost no inverse points. The further deduction that his generosity is delicate and abundant may be made from one sentence in the preface: "It was with pleasure that I received the approval of the Council of the Mathematical Association to arrange for the Association to receive one half of the royalties from the sale of this book." Members of the Association now have a simple but effective way of showing their gratitude.

T. A. A. B.

More Figures for Fun. By J. A. H. Hunter. Pp. ix, 118. (Phoenix House)

This book, on the same lines as the author's *Figures for fun*, reviewed in the Gazette, Vol. XLII, No. 341, p. 238, contains a further selection of 150 arithmetical puzzles.

T. A. A. B.

Gödel's Proof. By E. Nagel and J. R. Newman. Pp. 118. 12s. 6d. 1959. (Routledge and Kegan Paul)

Gödel proved in 1931 that every formal system rich enough to contain arithmetic necessarily contains formulae which are neither provable nor refutable in the system. The methods Gödel used have had immense influence in all subsequent work in mathematical logic, and the object of this account is to give to non-mathematical readers some idea of what these methods are—the mathematician who wants to get to the core of the subject must of course look elsewhere.

R. L. Goodstein

Notices d'Histoire des Mathematiques. By HENRI LEBESGUE. (Monographies de *L'Enseignement Mathematique*, no. 4.) Pp. 116. Fr. Suisses 16. 1958. (Institut de Mathematiques, Universite, Geneve)

Lebesgue is not usually associated with the history of mathematics, and he did not think of himself as an historian. But he did write several essays on mathematicians of the past—Vieta, Vandermonde, C. Jordan, R-L. Baire, A-M. Ampere—and these essays have been gathered together, along with an article on mathematicians of the College de France and some extracts from correspondence, to form this little book. As is often the case with outstanding exponents of a branch of mathematics or science, Lebesgue writes with a depth of insight into the work of his predecessors that the professional historian must acknowledge with gratitude.

M. A. HOSKIN

Mathematics Before Newton: an inaugural lecture given in the University College of Rhodesia and Nyasaland. By A. R. MANWELL. Pp. 57. 4s. 1959. (Oxford University Press)

This unpretentious little work is sensibly and unashamedly based on good secondary sources, and it contains what is probably the best available brief summary of both pure mathematics and mathematical physics from ancient to early modern times.

M. A. HOSKIN

History of Mathematics. By D. E. SMITH. Vol. 1, pp. xxii, 596; Vol. 2, pp. xii, 725. 2 vols., £2. This edition, 1958. (Dover Publications. London: Constable)

Since its first appearance thirty-five years ago, Smith's *History* has been a favourite for its attractive presentation and innumerable illustrations. It is a mine of information, pleasant to read and an ideal purchase for a school library. Inevitably the lapse of time has rendered important sections out of date, but for once this hardly seems to matter: it is a book for those who read for enjoyment as well as information, and this reprint in soft covers is welcome.

M. A. HOSKIN

History and Philosophy of Science: An Introduction. By L. W. H. HULL. Pp. xii, 340. 25s. 1959. (Longmans)

This book contains an outline of the history of science from earliest times to the present day. It is simply written, but it deals with real problems, and the occasional sections on philosophers whose thought influenced the development of science are particularly welcome. There are important omissions, particularly in the treatment of the biological sciences, but the work as a whole is sound and one to be recommended to undergraduates and the better sixth-formers.

M. A. HOSKIN

Mathematics and Logic for Digital Devices. By J. T. CULBERTSON. Pp. 224. 36s. 1958. (D. van Nostrand Co., Ltd., London)

Mathematicians may find it hard to believe that the advent of digital computers has created a market for books on binary arithmetic, but this is certainly the case, as the number of vain requests which the Association Library receives testifies. This book competently fills the gap, and equally competently discusses the elements of Boolean Algebra, with applications to switching circuits.

R. L. GOODSTEIN

Aufgabensammlung zu den Gewöhnlichen und Partiellen Differentialgleichungen. By G. HOHEISEL. Pp. 124. DM. 2.40. 1958. (W. de Gruyter, Berlin)

This collection of examples with solutions is a supplement to the author's informative little books on ordinary and partial differential equations (Sammlung Göschen, Bd. 920, 1003).

R. L. G.

Table for the Solution of Cubic Equations. By H. E. SALZER, C. H. RICHARDS and ISABELLE ARSHAM. Pp. 161. 58s. 1958. (McGraw-Hill, London)

The table gives the values of the functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ to 7 decimal places with first and second differences, where

$$(2\theta)^{\frac{1}{3}} f_1(\theta) = \{1 + \sqrt{(1 + 4/27\theta)}\}^{\frac{1}{3}} + \{1 - \sqrt{(1 + 4/27\theta)}\}^{\frac{1}{3}},$$

$$f_2(\theta), f_3(\theta) = -\tfrac{1}{2}f_1(\theta) \pm \sqrt{\{\tfrac{1}{4}f_1(\theta)^2 - 1/\theta f_1(\theta)\}},$$

$\theta = ac^2/b^3$, and the roots of $ax^3 + bx + c = 0$ are $-(c/b)f_r(\theta)$, $r = 1$, 2, 3, θ and $1/\theta$ both being tabulated from $-1{\cdot}000$ to $+1{\cdot}000$ by steps of $\cdot 001$.

R. L. G.

Formale Logik. By P. LORENZEN. Pp. 164. DM 4.80. 1958. Sammlung Göschen. (Walter de Gruyter, Berlin)

Due to the very condensed nature of much of the material and the emphasis on the operative treatment of logic as opposed to the more usual axiomatic treatment of an introductory text, this is not a very suitable book for undergraduate study.

The author starts with the usual linguistic considerations followed by a modern account of the syllogistic modi using the axiomatic method. The system of axioms used consists of basic implications and basic rules. The rules are of the form: if ... then ... . These rules are to be taken as *imperative*. The author completes the axiomatic treatment of classical logic by defining 9 more connectives and resorting to the truth table technique for showing the completeness of the list of 10 connectives.

The next part deals with calculi. A calculus is a system of basic rules and atomic figures and a proof is a sequence of *Schematic operations* with the figures of a calculus. As the axiomatization of the logic of

connectives uses only basic rules of the form, "if A_1 and A_2 and ... then A", this logic can be conceived as a calculus. A Gentzen type calculus is next made the object of further study and its consistency and completeness are proved.

Gödel's incompleteness theorem is stated as well as Skolem's theorem on pathological models of incomplete axiomsystems of theories. The logic of equality (the ι operator, abstraction, relations and functions) completes the book.

K. L. STEWART

The Principles of Science. By W. STANLEY JEVONS. Pp. 769. 24s. 1958. (Dover Publications, New York)

The *Principles of Science* (first published in 1873) is one of the classical expositions of the traditional belief in two distinct but parallel kinds of inference-theory, deductive and inductive logic. The book shows a considerable advance on J. S. Mill's theories of scientific reasoning, but is by no means as modern in tone as the new introduction by Professor Nagel would suggest; for even though Jevons had a much closer aquaintance with actual scientific procedures than Mill did, he only partially grasped the importance of models and grossly over-estimated the possibilities of the application of probability to the problems of induction. This is not a book for students for it is very long and its curiously evangelical style makes it difficult to read. However it should not be overlooked by those interested in the history of logic or the development of the philosophy of science.

R. HARRÉ

Structures Algébriques et Structures Topologiques. Monographies de L'Enseignement Mathématique, No: 7. Pp. 200. 20 fr.s. 1958. (Institut de Mathématiques, Université, Geneva)

Twelve of France's leading mathematicians cooperated in a series of lectures at the Institut Henri-Poincaré in Paris in 1956 and 1957, and 17 of their lectures are reprinted in this volume. H. Cartan contributes lectures on algebraic structures and dimension theories; G. Choquet on vector spaces, linear forms, the number system and the structure of metric spaces; P. Dubreil on rings, congruences and ideals; A. Lichnerowicz on linear mappings and matrices; A. Revuz on projective, Euclidean and metric spaces; Ch. Pisot on general topology and compact and locally compact spaces; P. Lelong on quadratic and hermitian forms; L. Lesieur on the classical groups; J. Dixmier on function spaces and convergence; R. Godement on groups and topological vector spaces; J-P. Serre and L. Schwartz on homotopy and homology theory.

The lectures are related by many common threads which help to carry the reader's attention and interest and furnish a unique and stimulating introduction to many branches of modern mathematics.

R. L. GOODSTEIN

Einführung in Theorie und Anwendungen der Laplace-Transformation. By G. DOETSCH. Pp. 301. Fr./D.M. 39.40. 1958. (Birkhäuser, Basel und Stuttgart)

This is an account of those properties of the Laplace-Transformation needed for its applications to the solution of differential equations; it is intended for students of mathematics, physics and engineering: to the latter it will give all they are likely to need, for the former class it presents a clear and rigorous introduction to the subject. The material, inevitably, is all contained in the comprehensive "Handbuch" of the author; but is not a summary of that work, since the treatments of many subjects is different and new results on some problems in the applications have been added. The book is well-written and beautifully produced.

J. L. B. COOPER

Theory of Approximation. By N. I. ACHIESER. (Translated by C. J. HYMAN) Pp. x, 307. 40s. 1958 (Frederich Unger, N.Y.; Constable, London)

The theory of the best possible approximation to functions of a real variable by functions of a given type is a classical branch of mathematics to which Russian mathematicians from Tchebysheff onwards to Bernstein and the author of the present book have made great contributions. The book gives a clear and elegant exposition of the subject, using the methods of functional analysis to a considerable extent. Chapter I deals with approximation in normed linear spaces and Hilbert Space, including Müntz' theorem on completeness of a system of powers. Chapter II deals with the Tchebysheff theory of best approximation by polynomials; Chapter III is an introduction to the theory of Fourier series and integrals, and is followed by a discussion of extremal properties of integral functions of exponential type; in Chapter IV are theorems connecting differentiability properties of functions with their best approximations by trigonometric sums. Chapter VI deals with Wiener's Tauberian theorem, giving a proof based on the theory of normed rings but apparently earlier and certainly less elegant than Gelfand's.

The book closes with additional notes, a set of problems, notes on the literature and an index.

Apart from a few minor defects, such as the use of "normalised" for "normed" space, "Tauber" for "Tauberian" theorem, the translation is well done.

J. L. B. COOPER

BRIEF MENTION

Theory of Dielectrics. By H. FRÖHLICH. 2nd Ed. Pp. 192. 30s. 1958 (Oxford University Press)

Additional material in the second edition includes the general theory of the static dielectric constant.

An Introduction to Probability Theory and its Applications. Vol. I. By W. FELLER. 2nd Ed. Pp. 461. 56s. 1957. (Chapman & Hall)

A new chapter on combinatorial methods has been added in the second edition.

Numerical Analysis. By D. R. HARTREE. 2nd Ed. Pp. 302. 42s. 1958 (Oxford University Press)

The main change from the first edition is in the Chapter on the numerical integration of differential equations. The review of the first edition is in vol. XXXVII, p. 309.

Mathematics in Business. By L. L. LOWENSTEIN. Pp. 364. 40s. 1958 (Chapman & Hall)

A first course in the mathematics of business covering percentages, profit and loss, simple and compound interest, annuities and logarithms.

Conformal Transformations in Electrical Engineering. By W. J. GIBBS. Pp. 219. 45s. 1958 (Chapman & Hall)

A swift progression from the elements of Field Theory (without vectors) and complex functions to transformations involving elliptic functions.

Atomic Energy Levels. Vol. III. Pp. 283. \$2.50. 1958. **Annual Report 1957 of the National Bureau of Standards.** Pp. 143. 45 cents. 1958. **Further Contributions to the Solution of Simultaneous Linear Equations and the Determination of Eigenvalues.** Pp. 81. 50 cents. 1958

Issued by the U.S. Department of Commerce, the first two of these note books are of interest chiefly to Physicists. The third contains papers by Stiefel on Linear Algebra, by Henrici on the quotient difference algorithm and by H. Rutishauser on the solution of eigenvalue problems by the so-called LR-transformation.

Subtabulation. Pp. 54. 7s.6d. 1958 (H.M. Stationery Office)

This is a companion booklet to the volume *Interpolation and Allied Tables* prepared by H.M. Nautical Almanac Office.

Collected Papers of Prof. R. Vaidyanathaswamy. Pp. 589. 15 Rs. 1958. (University of Madras)

The papers cover an impressively wide range of subjects, from algebraic geometry and the theory of numbers to symbolic logic.

THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

Change of address should be notified to the Membership Secretary, Mr. M. A. Porter. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

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BRANCH REPORTS

NOTTINGHAM AND DISTRICT BRANCH OF THE MATHEMATICAL ASSOCIATION

Report for the year 1958–59.

The Annual General Meeting was held on 6th. December 1958 at the Institute of Education, Nottingham University. The reports of the Secretary and Treasurer were received and officers elected for the year. A vote of thanks was expressed in appreciation of the work of the Secretary, Mr. F. Chettle, who was leaving Nottingham. Mr. L. Prior of Nottingham University spoke on 'The Projection of Conics into Circles'. Various examples were given in which a real result was expected but turned out to be an imaginary figure. It was apparent that the speaker had shaken certain complacency concerning the projection of conics into circles, especially at sixth form level.

The Spring Term meeting on 14th. March was organised as a one-day conference held at Nottingham University. The speaker at the morning session was Mr A. P. Rollett who had some interesting and helpful comments to make on the recent Ministry publication 'The Teaching of Mathematics in Secondary Schools'. The first afternoon session was addressed by Mr. A. L. Brown of Nottingham University on the intriguing topic 'Dido's Domain'. Mr. Brown explained that Dido could claim an area of land in Carthage by enclosing it with a bull's hide. He showed that the area of a circle is greater than any isoperimetric curve and then discussed various branches of Mathematical Physics which use similar conditions. This was a very interesting and novel lecture. Dr. Woolley of Nottingham University then spoke on 'The Expanding Universe'. Dr. Woolley's topic introduced recent research methods and differences of opinion between the experimental and theoretical Physicists in this modern field of research in which radio-astronomy will play a large part in supplying the answers.

The success of the previous one-day conferences led to a change in the Summer meeting this year when a similar conference was held at Nottingham University on Saturday, 13th. June. This meeting was designed to interest all present members, yet at the same time be attractive to all teachers of Mathematics in the various types of Secondary Schools. This form of meeting was again successful, the attendance being over sixty, in number. The Speaker at the morning session was Miss Y. B. Giuseppi of the Dick Sheppard School, London Miss Giuseppi's topic was 'A Fresh Start in the Comprehensive School' in which many problems and experimental ideas were introduced and discussed. This was an excellent address which was

received with enthusiasm by the audience. After lunch, the first speaker was Mr. J. T. Combridge of King's College, London, this being his second visit to the Nottingham Branch. Mr. Combridge spoke on 'Mathematics—for the few or many?' in which he outlined the necessity for discovery and enjoyment instead of the highly unimaginative way in which Mathematics is taught in some of our schools. For the benefit of non-members present Mr. Combridge discussed some of the activities of the Mathematical Association, which was a very useful contribution to the day's proceedings. The second speaker at the afternoon session, Dr. Z. P. Dienes of Leicester University explained that the title of his address 'Learning Mathematics through Experience' was based on experiments which he is carrying out in Junior Schools aiming to break down the barriers in order to make Mathematics enjoyable for all. The speaker showed some of the games he had devised for experience in the mathematical processes. This address was enthusiastically received and it is hoped that Dr. Dienes will report on the progress of his experiments at a later meeting.

The Officers for the year were:

President:	Dr. G. Power.
Vice-President:	Mr. K. R. Imeson.
Secretary:	Mr. H. L. W. Jackson.
Treasurer:	Mr. C. R. Swaby.

H. L. W. Jackson
(Hon. Sec)

EXETER BRANCH OF THE MATHEMATICAL ASSOCIATION

Report for the year 1959.

The Branch has now just over 60 members of whom 28 belong to the main Association. There have been five meetings held during the year in the Library of the Institute of Education and the details are as follows:

February 6th 1959.	A Discussion introduced by Mr. A. P. Rollett of the Ministry of Education's pamphlet on The Teaching of Mathematics in Secondary Schools.
March 6th 1959.	Mr. A. R. Pargeter of Taunton's School, Southampton, spoke on Plaited Polyhedra.
May 1st 1959	Mr. W. Armistead of Christ's Hospital gave a lecture on the Geometry of the Cycloid.

June 12th 1959. Professor D. Rees of Exeter University gave a lecture on Impossible Problems.

December 3rd 1959. Mr. C. G. Nobbs of the City of London School spoke on The Reform of Geometry Teaching—the next stage.

Meetings have been well attended by members and we have been pleased to welcome students from the University, St. Luke's Training College and VIth forms from schools.

Officers for the Session have been as follows:

President:	MR. A. P. ROLLETT
Vice-President:	PROFESSOR T. ARNOLD BROWN
Treasurer:	MISS L. G. BUTTON
Secretary:	MISS N. A. COMERFORD.

NORA A. COMERFORD (Hon Sec.)

CARDIFF BRANCH OF THE MATHEMATICAL ASSOCIATION

Report for the year 1959–60

OFFICERS:

President: MR. I. E. HUGHES.
Treasurer: MR. R. A. JONES.
Secretary: MR. W. H. WILLIAMS.

Meetings:

Monday, 12th. October: The retiring chairman, Dr N. D. Hayes, gave a stimulating address on "Mathematics in Industry", in which he touched on computing, linear programming, statistics, and operational research, and dealt in detail with what he termed the Diet Problem. Saturday, 21st. November: A discussion, initiated by Miss M. E. Jones, and Mr. W. H. Williams, was held on "The aims and contents of the Mathematics Course". There were several contributions and suggestions from those present and certain matters were referred later to the Welsh Joint Secondary Committee.

Monday, 25th. January: Dr. C. Rogers, gave a very valuable lecture on "The Digital Computer: What it is and what it does". It led to many questions and observations.

Monday, 7th March: The Branch was very pleased to have a visit from a former President of the Association, Mr. W. Hope-Jones, who gave in his own inimitable way a talk on "Probability and Expectation".

W. H. WILLIAMS (Hon. Sec.)

BOOKS FOR REVIEW

ABBOTT, P. and C. E. KERRIDGE. **National Certificate Mathematics. Vol. I—1st yr. course.** Revised by W. E. FISHER. Pp. 410. 1960. 9s. 6d. (English Universities Press Ltd.)

ALEXITS, G. **Konvergenzprobleme der Orthogonalreihen.** Pp. 307. 1960. (Hungarian Academy of Sciences)

ARTIN, E. **Galoissche Theorie.** [Mathematisch-Naturwiss. Bibl. Bd.28.] Pp. 86. 1959. DM5,30. (Teubner, Leipzig)

ARZÉLIÈS, HENRI. **Milieux Conducteurs ou Polarisables en Mouvement. Études Relativistes.** Pp. xliv + 347. 1959. 58 NF. (Gauthier-Villars)

ASSER, G. **Einführung in die Mathematische Logik. Teil I.** [Mathematische Naturwiss. Bibl. Bd.18.] Pp. 184. 1959. DM11,25. (Teubner, Leipzig)

ATKIN, R. H. **Classical Dynamics.** Pp. 273. 1960. 30s. 0d. (Heinemann, London)

BACHET, CLAUDE-GASPAR. **Problemes Plaisants et Delectables.** 5th Ed. Pp. 243. 1959. 9 NF. (Albert Blanchard, Paris)

BALL, W. W. ROUSE. **String Figures (and other Monographs).** 3rd Ed. Pp. 489. 1960. $3.95. (Chelsea Publishing Co., New York)

BLAKEY, J. **Intermediate Pure Mathematics.** 2nd Ed. Pp. 458. 1960. 21s. 0d. (Cleaver-Hume Press Ltd.)

BOCHNER, SALOMON. **Lectures on Fourier Integrals.** Translated by MORRIS TENENBAUM and HARRY POLLARD. [Annals of Mathematics Studies No. 42.] Pp. 333. 1960. 40s. 0d. (Princeton University Press: London, Oxford University Press)

BOURBAKI, N. **Elements de Mathematique. Fascicule XXVI. Groupes et Algebres de Lie. Chapter I—Algebres de Lie** Pp. 148. 1960. 21 NF. (Hermann, Paris)

BRODETSKY, SELIG. **Memoirs from Ghetto to Israel.** Pp. 323. 1960. 21s. 0d. (Weidenfeld & Nicholson, London)

BURNS, P. F. **Daily Life Mathematics—Book V.** Pp. 340. 1959. 15s. 6d. (Ginn and Co., Ltd.)

BURNSIDE, W. **Theory of Probability.** Pp. 106. 1960. $1.00. (Dover Publications, Inc., New York)

BUSH, ROBERT B. and W. K. ESTES. **Studies in Mathematical Learning Theory.** Pp. 432. 1960. 92s. 0d. (Stanford University Press: London, Oxford University Press)

CASSELS, J. W. S. **An Introduction to the Geometry of Numbers.** Pp. 344. 1959. (Springer-Verlag, Berlin)

CHESTERMAN, H. See HOLLAND, D. A.

CHURCHILL, R. V. **Complex Variables and Applications.** 2nd Ed. Pp. 297. 1960. 52s. 6d. (McGraw-Hill Publishing Co. Ltd.)

CLARKE, L. H. **A General Certificate Calculus.** 2nd Ed. Pp. 254. 1960. 10s. 6d. (Heinemann Ltd.)

CLARKE, L. H. **Trigonometry at 'O' Level.** Pp. 150. 1960. 8s. 6d. (Heinemann Ltd.)

CRAIG, A. T. See HOGG, ROBERT V.

DEBREU, GERARD. **Theory of Value. An Axiomatic Analysis of Economic Equilibrium.** Pp. 114. 1959. 38s. 0d. (John Wiley, New York: Chapman & Hall Ltd., London)

DEDEKIND, R. **Was sind und was sollen die Zahlen?** Pp. 47. 1959. DM3,80. (Friedr, Vieweg & Sohn, Braunschweig)

DELACHET, ANDRE. **Les Logarithmes et leurs Applications.** Pp. 127. 1960. (Presses Universitaires de France, Paris)

DERMAN, C. and M. KLEIN. **Probability and Statistical Inference for Engineers. A First Course.** Pp. 144. 1960. 30s. 0d. (Oxford University Press)

DUSCHEK, A. and A. HOCHRAINER. **Grundzüge der Tensorrechnung in Analytischer Darstellung. Teil 1: Tensoralgebra.** Pp. 171. 1960. $5.70. (Springer-Verlag, Vienna)

FEENBERG, E. and G. E. PAKE. **Notes on the Quantum Theory of Angular Momentum.** Pp. 56. 1960. 10s. 0d. (Stanford University Press: London, Oxford University Press)

FLAVELL, J. S. and B. B. WAKELAM. **Primary Mathematics—An Introduction to the Language of Number.** Basic Book I. Pp. 128. 1960. 5s. 6d. [Also Teachers' Book—Pp. 44. 6s. 0d. (Methuen & Co. Ltd.)

FORSYTH, A. R. **Theory of Differential Equations.** Six volumes bound as three. 1960. $15.00 the set. (Dover Publications, New York)

GARNIER, R. **Cours de Mathematiques Generales. Analyse et Geometrie.** [Cours de la Faculte des Sciences de Paris—Tome IV.] Pp. 275. 1959. 45,00 NF. $9.46. (Gauthier-Villars)

GAZALE, MIDHAT J. **Les Structures de Commutation a m Valeurs et les Calculatrices Numeriques.** [Collection de Logique Mathematique, Serie A. Vol. XV.] Pp. 76. 1959. 14 NF. (Gauthier-Villars, Paris)

GOURSAT, E. **A Course in Mathematical Analysis, Vol. I. Applications to Geometry; Expansion in Series; Definite Integrals; Derivatives and Differentials.** Translated by E. R. HEDRICK. Pp. 548. 1960. $2.25. (Dover Publications Inc., New York)

GOURSAT, E. **A Course in Mathematical Analysis, Vol. II. Part I—Functions of a Complex Variable.** Pp. 259. $1.65. **Vol. II. Part II—Differential Equations.** Pp. 300. $1.65. Translated by E. R. HEDRICK and OTTO DUNKEL. 1960. (Dover Publications Inc., New York)

GRABBE, E. M., S. RAMO and D. E. WOOLRIDGE. Edited by—**Handbook of Automation, Computation and Control. Vol. II—Computers and Date Processing.** Pp. 1,005 + Index 37. 1959. 140s. 0d. (Chapman and Hall Ltd.: John Wiley, New York)

GREEN, S. See SCHWARTZ, M.

GRENANDER, ULF. Edited by—**Probability and Statistics.** [The Harold Cramér Volume.] Pp. 434. 1960. 100s. 0d. Almqvist & Wiksell, Stockholm: John Wiley, New York; Chapman and Hall, London)

HADAMARD, J. **Essai sur la Psychologie de L'Invention dans le Domaine Mathematique.** 1st Ed. in France. Translated from the English. Pp. 134. 1959. 8 NF. (Albert Blanchard, Paris)

HARDY, G. H. and E. M. WRIGHT. **An Introduction to the Theory of Numbers.** 4th Ed. Pp. 421. 1960. 42s. 0d. (Clarendon Press: Oxford University Press, London)

HART, W. J. **College Algebra and Trigonometry.** Pp. 466. 1960. 40s. 0d. (D. C. Heath & Co., U.S.A.; G. G. Harrap & Co. Ltd., London)

HASSER, N. B., J. P. LASALLE and J. A. SULLIVAN. **Introduction to Analysis. A Course in Mathematical Analysis, Vol. I.** Pp. 688 + xxxi. 1959. $8.50. (Ginn and Co., Boston, Mass.)

HERSEE, E. H. W. **A Simple Approach to Electronic Computers.** Pp. 104. 1959. 12s. 6d. (Blackie & Son Ltd.)

HOCHRAINER, A. See DUSCHEK, A.

HOGG, R. V. and A. T. CRAIG. **Introduction to Mathematical Statistics.** Pp. 245. 1960. 47s. 0d. (Macmillan & Co.)

HOLLAND, D. A. and H. CHESTERMAN. **Oxford Graded Arithmetic Problems.** 1A—Pp. 64. 1960. 2s. 6d. 2A—Pp. 64. 1960. 2s. 6d.

HORN, J. and H. WITTICH. **Gewöhnliche Differentialgleichungen.** [Goschens Lehrbücherei Band 10.] Pp. 275. 1960. (Walter de Gruyter, Berlin)

JAFFARD, P. **Les Systemes d'Ideaux.** Pp. 132. 1960. 25 NF. (Dunod, Paris)

JAMES, E. J. **Mathematical Topics for Modern Schools.** First Year—Bks. 1, 2, 3. Second Year—Bks. 1, 2, 3. Pp. 16 each book. 1960. 1s. 6d. each book. (Clarendon Press: Oxford University Press, London)

JEGER, MAX. **Konstruktive Abbildungsgeometrie.** [Einzelschriften zur Gestaltung des Mathematisch-Physikalischen Unterrichtes, Heft 1.] Pp. 79. 1959. (Verlag Räber & Cie, Luzern)

JULIA, GASTON. **Elements d'Algebre.** Pp. 207. 1959. 38 NF. (Gauthier-Villars)

KAC, MARK. **Statistical Independence in Probability, Analysis and Number Theory.** [The Carus Mathematical Monographs No.12 Pp. 93. 1959. 245. 0d. (The Mathematical Association of America: Wiley, New York; Chapman & Hall, London)

KARLIN, S. **Mathematical Methods and Theory in Games, Programming and Economics.** Two volumes. **Vol. I Matrix Games, Programming and Mathematical Economics.** Pp. 433. 1959. 75s. 0d. **Vol. II Theory of Infinite Games.** Pp. 386. 1959. 75s. 0d. (Pergamon Press Ltd.)

KEMENY, J. G. and J. L. SNELL. **Finite Markov Chains.** 1960. 37s. 6d. (D. Van Nostrand Co. Ltd.)

KERRIDGE, C. E. See ABBOTT, P.

KLEIN, M. See DERMAN, C.

KLINE, M. **The Language of Number.** Bk. I. Pp. 151. 1960. 7s. 6d. (G. G. Harrap & Co., Ltd.)

LANGFORD, C. H. See LEWIS, C. I.

LASALLE, J. P. See HASSER, N. B.

LAVOINE, J. **Calcul Symbolique, Distributions et Pseudo-Fonctions.** [B-Methodes de Calcul II.] Pp. 110. 1959. 10 NF. (Centre National de la Recherche Scientifique, Paris)

LAWDEN, D. F. **A Course in Applied Mathematics.** Vols. 1 and 2 [bound together]. Pp. 655. 1960. 70s. 0d. (The English Universities Press Ltd.)

LEWIS, C. I. and C. H. LANGFORD. **Symbolic Logic.** Pp. 518. 1960. $2.00. (Dover Publications, Inc., New York)

LUCAS, E. **Recreations Mathematiques.** 2nd Ed. Pp. 254. 1960. 8 NF. (Albert Blanchard, Paris)

LUKACS, E. **Characteristic Functions.** Pp. 216. 1960. 38s. 0d. (Chas. Griffin & Co., Ltd.)

MENNINGER, K. **Zwischen Raum und Zahl. Mathematische Streifzüge.** Pp. 222. 1960. (Ullstein Taschenbücher-Verlag GmbH., Frankfurt/M.)

MESERVE, B. E. See ROSENBACH, J. B.

MILNE-THOMSON, L. M. **Theoretical Hydrodynamics.** 4th Ed. Pp. 660. 1960. 65s. 0d. (Macmillan & Co., London)

MOORE, J. T. **Fundamental Principles of Mathematics.** Pp. 630. 1960. $7.00. (Rinehart & Co., New York)

MOSS, G. A. **Geometry for Juniors.** Bks. 1, 2, 3, 4. Pp. 36 each book. 1960. 3s. 6d. each book. Also Teachers' Book. (Basil Blackwell, Oxford)

MUNIR, D. G. **Common Entrance Arithmetic and Algebra.** Parts I and II—complete with answers. Pp. 269. 1960. 12s. 6d. Also without answers or each Part separately. (Methuen & Co.)

PAKE, G. E. See FEENBERG, E.

PIERPONT, J. **Functions of a Complex Variable.** Pp. 583. 1960. $2.45. (Dover Publications, Inc., New York)

PIERPONT, J. **The Theory of Functions of Real Variables.** Vol. I—Pp. 560. 1960. $2.45. Vol. II—Pp. 645. 1960. $2.45. (Dover Publications Inc., New York)

PRIWALOW, I. I. **Einführung in die Funktionentheorie, Teil III.** [Mathematische-Naturwissenshaftliche Bibl. 23.] Pp. 188. 1960. DM12,90. (Teubner, Leipzig)

ROSENBACH, J. B., E. A. WHITMAN, B. E. MESERVE and P. M. WHITMAN. **Intermediate Algebra for Colleges.** 2nd Ed. Pp. 307 + xxxi. 1960. (Ginn & Co., London)

RUTLEDGE, W. A. See SCHWARTZ, M.

SAUER, R. **Ingenieur-Mathematik. Vol. I—Differential-und Integralrechnung.** Pp. 304. 1959. DM24,0. (Springer-Verlag, Berlin)

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